Numerical Solutions of the Klein-Gordon Equation with Adaptive Mesh Refinement

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***Resumen*— Se presenta la evolución numérica de un campo escalar de prueba en el espacio de Minkowski, utilizando la técnica de mallas refinadas adaptativas (AMR). La dinámica del campo escalar está dada por la ecuación de Klein Gordon con un potencial exponencial, el cual es usado como un modelo de campos escalares tipo quintaesencia. Primero se realizará la descripción del algoritmo AMR. Luego se realizará un análisis relacionado con la convergencia de las simulaciones numéricas, encontrando convergencia de segundo orden, las cuales son consistentes con el esquema de diferencias finitas de segundo orden usado.**

Palabras clave*— *Ecuación de Klein Gordon, mallas refinadas adaptativas, campo escalar, algoritmo, simulaciones numéricas

***Abstract*— In this paper we present the numerical evolution of a test scalar field on a Minkowski background using adaptive mesh refinement techniques (AMR). The Dynamics of the scalar field is given by the Klein Gordon equation with an exponential potential, which has been used as a model of quintessence scalar fields. As a first step in this work a description of the AMR algorithm is presented. Then we perform an analysis related to the convergence of the numerical simulations, founding convergence of second order, which is consistent with the second order finite difference scheme used.**

Key Word — Klein Gordon Equation, Adaptive mesh refinement, scalar field, algorithm, numerical simulations.

I. INTRODUCTION

Scalar fields have been of great interest since them have played an important role in general relativity and cosmology. For instance, scalar fields have been used in cosmology to give a possible solution to the horizon and flatness problems, by assuming a mechanism that provides an exponential cosmological growth [1]. Moreover, in [2], by considering the supernovae redshift observations, the authors explored the possibility of a cosmic scalar field to play the role of dark energy. Scalar fields have been also proposed as a model of dark matter, either to galactic [3] or cosmological scales [4]. On the other hand, AMR methods are important because they allow studying complex problems with a high accuracy, without increasing the computational cost too much. This is because when we solve the problems numerically these methods are adapted to the dynamics of the problem, using meshes that have different resolutions in different regions and that also adapt over time. In our case, an algorithm based on the work of Berger and Oliger [5], Berger and Colella [6] and Guzmán [7] was implemented with the aim of adapting the numerical solution to a specific one-dimensional problem as a first step to advance in the development of this type of algorithms.

In this work, we explore the evolution of a scalar field on a Minkowski space-time by using adaptive mesh refinement techniques. Specifically, in this first article, we focus the attention in the implementation of the AMR methods in our codes, in order to solve the Klein-Gordon equation with exponential potential, which is written as a first order system of equations by using the 3+1 formulation of the general relativity. It should be mentioned that the way this system of equations is written will allow us, in future works, to evolve the scalar field in a curve background and so be able to do a full 3D numerical study of the accretion of scalar field dark matter on to a Kerr black hole with AMR techniques, which is very useful at the time of giving high resolution close to the black hole. It should be noted that exponential potential has been used in a great variety of works. For instance, this potential is considered as a possible model for quintessence, see [8]. Moreover, this potential arises naturally in the context of Kaluza-

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Klein theories, as well as in a variety of supergravity models [9].

The paper is organized as follows, in section 2 we show the Klein Gordon equation in the 3+1 formulation of the general relativity as well as the potential and initial profile for the scalar field. In section 3, we describe the adaptive mesh refinement method used to solve the Klein-Gordon on the Minkowski space-time. In section 4, we present the evolution in time of the scalar field and show the self convergence test in order to validate our results. Finally in section 5, we present some conclusions. It is worth mentioning that the units we assume in the paper are such that $G = c = 1$.

II. KLEIN GORDON EQUATION

In this paper we consider the classical nature of the scalar field, as assumed in SFDM and quintessence models, from an effective Lagrangian

$$\mathrm{L} = -\mathrm{R} + (\nabla\Phi)^2 + \mathrm{V}(\Phi), \qquad (1)$$

where R is the Ricci scalar of the space-time,$\Phi$the scalar field and $V(\Phi)$ its potential. The variation of such Lagrangian with respect to Φ reduces to the Klein-Gordon (KG) equation, which rules the evolution of the scalar field.

$$\square\Phi - \frac{\mathrm{dV}}{\mathrm{d}\Phi} = 0, \qquad (2)$$

where the D'Alambertian operator for a general space-time is $\square\Phi - \nabla^2\Phi = \left(1/\sqrt{-\mathrm{g}}\right)\partial_\mathrm{u}\left[\sqrt{-\mathrm{g}}\mathrm{g}^{\mathrm{uv}}\,\partial\Phi\right]$. Since we work on a fixed background space-time there is no need to vary the Lagrangian with respect to the metric, which would imply Einstein's equations. In terms of the variables of the $3+1$ splitting approach of general relativity [10], the KG equation can be written as a first order system of equations as follows

$$\partial_\mathrm{t}\Pi = \partial_\mathrm{i}\left(\beta^\mathrm{i}\Pi + \alpha\sqrt{\gamma}\gamma^{\mathrm{ij}}\Psi\Psi_\mathrm{j}\right) - \alpha\sqrt{\gamma}\frac{\mathrm{dV}}{\mathrm{d}\Phi}, \qquad (3)$$

$$\partial_\mathrm{t}\Psi_\mathrm{i} = \partial_\mathrm{i}\left(\frac{\alpha}{\sqrt{\gamma}} + \beta^\mathrm{j}\Psi_\mathrm{i}\right), \qquad (4)$$

$$\partial_\mathrm{t}\Phi = \frac{\alpha}{\sqrt{\gamma}}\Pi + \beta^\mathrm{j}\Psi_\mathrm{j}, \qquad (5)$$

Where $\Pi = \sqrt{\gamma}\left(\partial_\mathrm{t}\Phi + \beta^\mathrm{j}\,\partial_\mathrm{j}\Psi\right)/\alpha$ and $\Psi_\mathrm{j} = \partial_\mathrm{j}\Phi$ are new first order variables, $\alpha$is the lapse function, β^i the shift vector, γ_{ij} are the components of the spatial induced metric and $\gamma = \det\left(\gamma_{\mathrm{ij}}\right)$. It is worth mentioning that in this work, we restrict our AMR numerical calculations to the Minkowski space time. However in the way these equations are written allow us to run simulations in a curve fixed background [11].

In this work, we will consider the exponential potential, which is given by the expression

$$\mathrm{V}(\Phi) = \mathrm{V}_0\mathrm{e}^{-\alpha\Phi} \qquad (6)$$

Where V_0 and α are positive constants. Here we work in units for which $c = G = 1$. On the other hand, in order to solve the first order KG system of equations (*5*), we provide a scalar field initial profile similar to that described in [12] , which corresponds to a time-symmetric wave modulated by a Gaussian profile

$$\Phi(0,\mathrm{x}) = \mathrm{A}\cos(\mathrm{kx})\mathrm{e}^{\frac{-(\mathrm{x}-\mathrm{x}_0)^2}{\sigma^2}}$$

$$\Psi(0,\mathrm{x}) = \frac{\partial\Phi(0,\mathrm{x})}{\partial\mathrm{x}} \qquad (7)$$

$$\Pi(0,\mathrm{x}) = 0$$

where A and σ are the amplitude and width of this initial profile.

III. ADAPTIVE MESH REFINEMENT

In numerical analysis, the AMR method adapts the grid resolution according to the dynamics of the specific problem to be solved, building refined meshes in determined regions of the domain, which can appear and disappear as it becomes necessary. Those grids also can move in regards of a physical criterion. In this work, we have built a refined grid that displaces with the maximum value of the numerical error (for instance, sensitive regions or that present turbulence). At the beginning of the simulations, the algorithm creates a regular Cartesian grid that covers the whole domain, which is called base-grid. In the case of the Klein Gordon equation, the regions that present the maximum numerical error displace with a constant velocity, which makes easier to adapt the movement of the refined grids. In other cases is normally based on the estimation of the numerical error or in proper parameters of the system.

A. Coupled Evolution.

A very important aspect in the AMR methods is that the adaptive meshes have a coupled evolution, which consists in a singular grid with different resolutions in some parts of the domain, and no different grids evolving independently. In order to do this, is necessary to consider two aspects: the first one is that the values of the base-grid are replaced with the values of the son-grid for each time step. This guarantees that the base-grid does not evolve to the next step with its numerical error (which is bigger). The second aspect is that the border points of the son-grid need to be calculated from the interpolation of the points in the base-grid at each time of the simulation. These can be better explained with Figure 1.

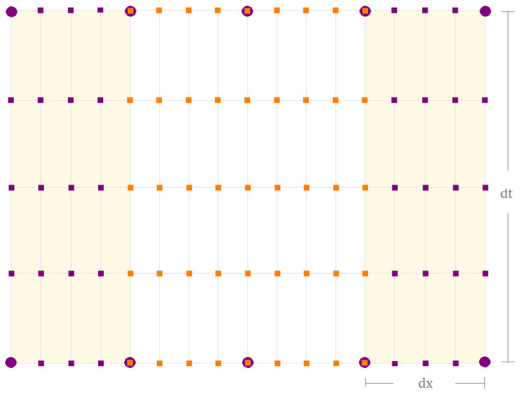

Figure 1 Evolution of a one-dimensional grid with a sub-domain refined by $\alpha = 4$. In this figure we represent the evolution of a single base-grid time step, where the circled violet points correspond to the base grid, the orange squares to the refined grid and the violet squares are calculated by using the interpolation of the base-grid in the shaded region. When the orange squares are inside the violet circles means that the values of the refined grid are replaced with the values of the grid-base. The time step of the base-grid is $dt = CFL * dx$, and the time step of the son-grid is $dt_c = CFL * dx_c = CFL * (1/\alpha) * dx$ $dt_c = CFL * (1/4) * dx = (1/4)dt$ and so on for more refined levels.

The algorithm discretizes the partial differential equations with the method of finite differences, integrating in time with the Runge Kutta schemes, and for the boundary calculations we impose out-going wave conditions, interpolating them with the Lagrange method. Finally, to adapt the grids with the dynamics of the problem, we displace them each certain amount of time steps, depending on the required velocity.

IV. NUMERICAL RESULTS

After carrying out the implementation of the AMR algorithm with the characteristics specified in the previous section, we solve the Klein Gordon equation (2) with an exponential potential as shown in equation ((6). In Figure 2 and Figure *3*, we plot the complete evolution for a time interval, where the base-grid and the refined-grid are displayed, the latter moving at the same speed as one of the propagated pulses, since the region near the pick of the pulse is the one that presents the maximum numerical error. Later, the numerical error associated with this solution is calculated, and it is shown in Figure 4, where it can be seen that the pulse traveling with a refined mesh presents a considerable decrease in the error. Finally, tests are carried out to verify the accuracy of the implemented algorithm, which in this case are the self-convergence and are presented in Figure *5*.

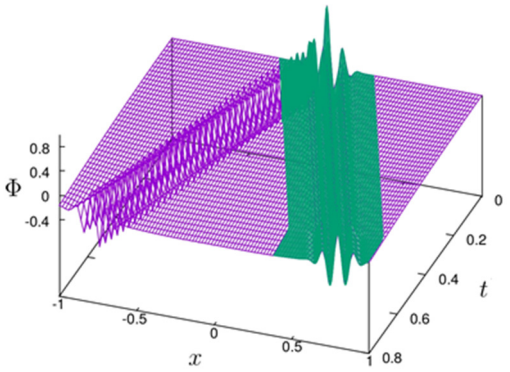

Figure 2 Evolution of the numerical solution for a complete time interval using AMR. The time range is $[0,0.8]$, the refined subdomain $[-0.3,0.3]$ moves at the same speed of one of the pulses within the base domain $[-1,1]$.

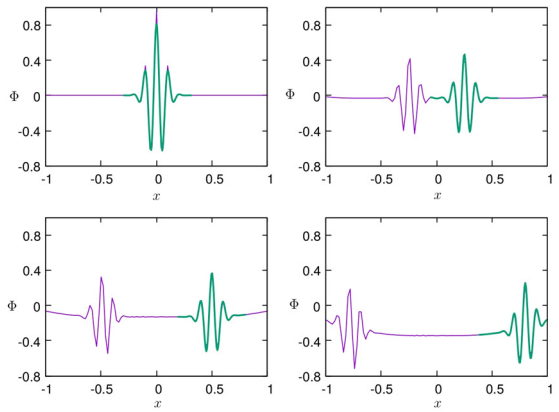

Figure 3 Captures for different instants of time for the evolution shown in Figure 2, $t = 0$, $t = 0.3$, $t = 0.6$ y $t = 0.8$ respectively.

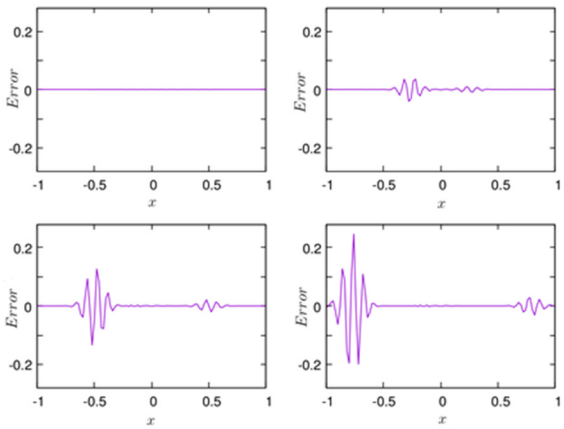

Figure 4 Evolution of the numerical error associated with each of the plots in Figure 3. The error is calculated by subtracting two numerical solutions with different base resolution.

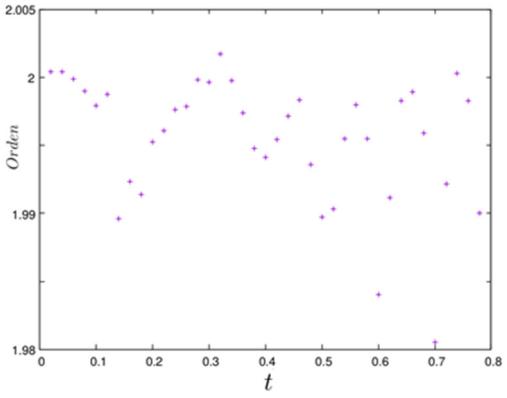

Figure 5 Convergence factor for each time step in evolution shown in Figure 2. This factor is calculated using three numerical solutions for three different base resolutions.

V. CONCLUSION

The same postulates used to develop this algorithm are extensible without problem to two and three dimensions, which are algorithms in process of development with favorable results. It was found that numerical error in a specific region can be reduced by refining only such region, reducing the computational cost. Moreover there is an adequate convergence by the AMR code, which is consistent with the second order finite differences used. Finally, we note that is not convenient to perform an abrupt refinement when improving the resolution considerably, it is better to do it by levels.

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