

ANALYTIC STUDY OF A PREY-PREDATOR MODEL WITH ALLEE EFFECT

Estudio analítico de un modelo predador-presa con efecto Allee

ABSTRACT

The Allee effect is introduced into intraspecific and two-species competitive models. We use phase plane analysis and simulation to investigate the dynamics of these models. Combining simple modeling and simulations, we demonstrate that the Allee effect alone could lead to alternative stable states in two-species competitive systems. In such systems, if interspecific competition is intense, both species may go extinct even if their population densities are high.

KEYWORDS: *Allee effect, prey-predator model.*

RESUMEN

El efecto Allee se introduce en modelos de competencia intraespecífica y dos especies en competencia. Se utiliza análisis de plano de fase y simulación para investigar la dinámica de estos modelos. Combinando modelación simple y simulación, se demuestra que el efecto Allee por sí solo podría llevar sistemas de dos especies en competencia a estados estables alternativos. En tales sistemas, si la competencia interespecífica es intensa, las dos especies podrían extinguirse aunque su densidad de población sea alta.

PALABRAS CLAVES: *Efecto Allee, Modelo predador-presa.*

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1. INTRODUCTION

The Allee effect, also called as 'negative competition effect' (Begon and Mortimer, 1981), refers to a decrease in per-capita growth rate at low population densities (Allee, 1931, 1938; Fowler and Baker, 1991; Burgman et al., 1993; Begon et al., 1996). The logistic model assumes that percapita growth rate declines monotonically with density. But for populations subject to an Allee effect, per-capita growth rate shows a humped curve increasing (from negative to positive) at low density, up to a maximum at intermediate density and then declining.

2. CONTENT

2.1 THE MODEL

Let us consider following prey--predator model

$$\begin{cases} \frac{dx}{dt} = rx \left(1 - \frac{x}{k} \right) - \frac{mx}{x+b} - \frac{qxy}{x^2+a} \\ \frac{dy}{dt} = sy \left(1 - \frac{y}{nx} \right). \end{cases} \quad (1)$$

In this model $x = x(t)$ represents the number of preys and $y = y(t)$ is the number of predators at a given moment t ($t \geq 0$). System (1) is a dynamical system in the plane $x - y$. This is a typical Lotka-Volterra model. Furthermore, the parameters have the following meanings:

- (1) r and s are the intrinsic growth rates or biotic potential of the prey and predators, respectively.
- (2) q is the maximal predator per capita consumption rate, i.e., the maximum number of prey that can be eaten by a predator in each time unit.
- (3) a is the number of prey necessary to achieve one-half of the maximum rate q .
- (4) n is a measure of the food quality that the prey provides for conversion into predator births.
- (5) k is the prey environment carrying capacity [8].
- (6) m and b are constants that indicate the severity of the Allee effect that has been modeled.

This model considers three aspects for describing the interaction:

(i) A logistic type growth function for predator;

(ii) A functional response of predators of nonmonotonic type; and

(iii) The Allee effect [5,9,11,24], which is considered in the prey growth function and occurs whenever fitness of an individual in a small or sparse population decreases as the population size or density also declines [5].

We shall study model (1) from the point of view of the general theory autonomous systems. First, we find equilibrium points of this system. These points are obtained by solving the algebraic system

$$\begin{cases} rx\left(1 - \frac{x}{k}\right) - \frac{mx}{x+b} - \frac{qxy}{x^2+a} = 0. \\ sy\left(1 - \frac{y}{nx}\right) = 0. \end{cases} \quad (2)$$

Second equation of system (2) gives

$$y = 0 \text{ or } y = nx.$$

2.2 COEXISTENCE OF PREYS AND PREDATORS

Let us suppose that $y > 0$ in (2). Then

$$y = nx,$$

i.e., the number of predators y increases proportionally to the number of preys x .

Now, we substitute equation $y = nx$ into the first equation of system (2) and after some algebraic calculations we arrive at the following fourth degree polynomial equation in the variable x :

$$rx^4 + (b-k)rx^3 + (ar+k(m+nq-br))x^2 + (bknq+a(b-k)r)x + ak(m-br) = 0. \quad (3)$$

Let

$$L = \frac{br-m}{s}. \quad (4)$$

First Case: $L = 0$.

We will assume that m and b are subject to

$$\begin{cases} bknq + a(b-k)r = 0. \\ m - br = 0. \end{cases}, \quad (5)$$

i.e.,

$$m = \frac{akr^2}{ar+knq} \text{ and } b = \frac{akr}{ar+knq}. \quad (6)$$

This assumption gives equation (8) in the form

$$rx^2 + (b-k)rx + (ar+k(m+nq-br)) = 0. \quad (7)$$

Taking into account equations (9a), equation (10) has the solution

$$x = \frac{k^2nqr}{2r(ar+knq)} \left(1 \pm \sqrt{1 - \frac{4(ar+knq)^3}{k^4n^2q^2r}} \right). \quad (8)$$

provided that

$$k^4n^2q^2r \geq 4(ar+knq)^3. \quad (9)$$

Observe that restrictions (6) imply that

$$0 < m < kr \text{ and } 0 < b < k. \quad (10)$$

For example, our assumptions hold if we set

$$a = 3, k = 10, n = 2, q = 2 \text{ and } r = 3. \quad (11)$$

This choice gives

$$b = 1.83673, m = 5.5102.$$

Thus, from (6) and (11) the corresponding equilibrium points are

$$(8.16327, 0), (4.65294, 9.30588) \text{ and } (3.51033, 7.02065).$$

The phase portrait near these points is shown in Figure 1. This graphic was made with the aid of *Maple 13*.

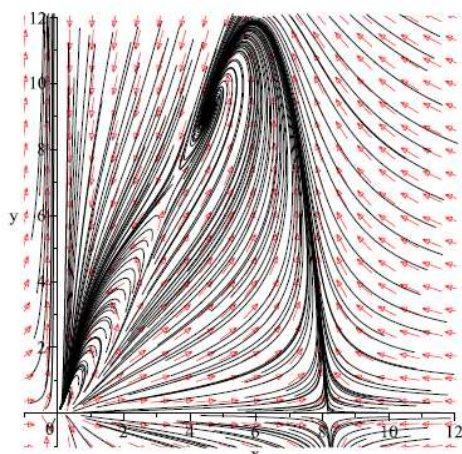


Figure 1

Another possible choice for the parameters is

$$b = \frac{k}{2}, a = \frac{k^2 nq}{2m}, r = \frac{2m}{k}, m > 16nq. \tag{12}$$

Second Case: $L < 0$

We choose

$$\begin{aligned} r &= \frac{(2a^2 + a + 1)nq}{a(a-1)}, \\ m &= \frac{(2a+1)^2 (a^2 + 1)nq}{2a^2(a-1)}, \\ k &= \frac{2a^2 + a + 1}{a}, \\ b &= \frac{2a^2 + a + 1}{2a}, \end{aligned} \tag{13}$$

where a is subject to the condition

$$a > \frac{1}{4}(7 + \sqrt{41}) \approx 3.35078.$$

Observe that

$$L = \frac{br - m}{s} = -\frac{2am}{(2a+1)^2 (a^2 + 1)s} < 0.$$

The advantage here is that our selection gives equation (7) in the form

$$(x-1)^2 (2ax^2 - (2a^2 - 3a + 1)x + 2a) = 0. \tag{14}$$

Solving equation (14) gives

$$x = \frac{1}{4a} \left(2a^2 - 3a + 1 \pm \sqrt{(a + 2a^2 + 1)(2a^2 - 7a + 1)} \right).$$

These data allow simulating the model.

3. CONCLUSIONS

We studied a prey-predator model with Allee effect. We also showed the way we may chose some of the parameters in a wide range in order to simulate the model. This helps in the understanding the way in which prey and predators may coexist.

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