Linear Algebra Concepts with SageMath for Systems Engineering students

Conceptos del Álgebra lineal con SageMath para estudiantes de Ingeniería de Sistemas

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***Abstract*— This work presents an innovative proposal for the teaching and learning of systems of linear equations, graphical interpretation in the plane, in space and orthogonal projection, in the subject of Linear Algebra, mediated by the SageMath software as a technology of learning and knowledge. The proposal was carried out in some Linear Algebra courses in the Systems Engineering career at the University of Cauca, helping to increase its quality, reduce the retention rate and contribute to educational innovation in the university student population. The methodology for the development of this work consisted of the design and application of activities that comprise a set of problems which require the creation of functions in the Python programming language under the SageMath, environment to obtain possible solutions and correlate algebraic and geometric representation registers of the mathematical objects necessary for its solution. We conclude at the end of this educational research the relevance of the integration of technological tools within the classroom in conjunction with active methodological strategies to stimulate the understanding of some concepts of Linear Algebra; since the student can visualize, manipulate and observe these abstract mathematical objects, eliminating unnecessary manual actions and focusing on the analysis of logical deduction for the solution of the proposed activities.**

***Index Terms*— ICT; educational innovations, algebra; geometry; equations**

***Resumen*— Este trabajo presenta una propuesta innovadora para la enseñanza y aprendizaje de sistemas de ecuaciones lineales, su interpretación gráfica en el plano, en el espacio y representación de la proyección ortogonal, en la asignatura de Álgebra Lineal, mediados por el software SageMath como una tecnología del aprendizaje y el conocimiento. Esta propuesta se llevó a cabo en algunos cursos de Álgebra Lineal en la carrera de Ingeniería de Sistemas de la Universidad del Cauca, contribuyendo a incrementar la calidad de este, disminuir el índice de retención y aportar a la innovación educativa en la población estudiantil universitaria. La metodología para el desarrollo de este trabajo consistió en el diseño y aplicación de actividades que comprenden un conjunto de problemas los cuales, en algunos casos, requieren la creación de funciones en el lenguaje de programación Python bajo el ambiente de SageMath, para obtener posibles soluciones y correlacionar registros de representación algebraico y geométrico de los objetos matemáticos necesarios para su solución. Concluimos al término de esta investigación educativa la pertinencia de la integración de herramientas tecnológicas dentro del aula de clase en conjunción con las estrategias metodológicas activas para estimular la compresión de algunos conceptos del Álgebra Lineal; ya que el estudiante puede visualizar, manipular y observar estos objetos matemáticos abstractos eliminando acciones manuales innecesarias y centrándose en el análisis de la deducción lógica para la solución de las actividades planteadas.**

***Palabras claves*— Álgebra, ecuación, geometría, innovación educativa, TIC.**

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I. INTRODUCTION

In the academic programme of Systems Engineering at the Universidad del Cauca, we find as a requirement the subject Linear Algebra in the second semester (Unicauca, 2022). This initiative aims to respond to the curricular reflection in the Faculty of Electronics and Telecommunications Engineering at the University of Cauca, which aims to improve the teaching and learning processes of undergraduate programmes in order to increase their quality, reduce retention rates and contribute to innovation in university education, as stated by (Maya & Pino, 2019).

The high retention rates in this subject have been evidenced in other universities, motivating studies that determine this low performance as shown in the article (Arias, Manco, & Uzuriaga, 2010); likewise, strategies have been proposed to solve difficulties due to the formalism of linear algebra through geometry as in (Dorier, Robert, Robinet, & Rogalsiu, 2000); or on the other hand, due to the teaching experience, also, deficient previous knowledge of the students in the subjects that demand the handling of some topics of Linear Algebra for its optimal development is observed, due to the level of complexity in the formalism and the connection between algebraic and geometric thinking in the teaching and learning of this as it is evidenced in: (Oktac, Sierpinska, & Anadozie, 2002) involving theoretical thinking, (Ortega, 2002) using computer systems of algebraic calculation, (Oktac & Trigueros, 2010) involving APOE theory, (Betancourt, 2014) including digital technologies, (Martinez & Vanegas, 2021) presenting a didactic sequence based on Van Hiele's model.

The aforementioned motivated to give a proposal to develop educational innovation actions as proposed by (Leal, Rojas, Ortiz, & Monrroy, 2020) which is achieved through the transformation of academic activities and not only the incorporation of new resources so "innovating means opening horizons, generating an investigative interest, enjoying the pleasure of inquiring, discovering, proposing, evaluating, but above all of inventing. It is about advancing in a critical stance towards existing postulates, because only when it becomes a subject of reflection, research and questioning is it possible to innovate" (Salcedo, 2016, as cited in Leal, Rojas, Ortiz, & Monrroy, 2020) on the other hand "despite the ethereal nature of the term, pedagogical innovation alludes to the systematisation and recognition of transformative practices and as such is an opportunity to make visible and recognise pedagogical practices according to the needs of the context" (Gómez 2016, as cited in Leal, Rojas, Ortiz, & Monrroy, 2020).
This is how the incorporation of the mathematical software SageMath (SageMath, 2022) is proposed, of which we have the manual (Bard, 2014) and for the specific application of this software to Linear Algebra with the book (Hefferon, 2021).
We consider this SageMath software as a TAC (Technology of Learning and Knowledge) since we intend to guide and focus the Information and Communication Technology (ICT) for educational uses together with an active methodology in order to improve the teaching-learning process in the classroom as suggested by (Unir, 2021), (Lozano, 2011), (Moya, 2013), corresponding to the need raised in the conclusions of the conference (Maya & Pino, 2019) to innovate in Engineering education and attending to multidisciplinarity from the teaching-learning of mathematics and in particular for the study of some topics of the subject Linear Algebra, such as solutions of systems of linear equations and geometric interpretation in the plane and in three-dimensional space, given the difficulties of students to face the study of these topics as stated in (Rodriguez, 2011), (Avilez, Romero, & Vergara, 2016), (Rosales, 2010).
SageMath is a free program, which can be used as an online or desktop application, and also allows working with the Python programming language, which is used in some of the activities of this project at a basic level, to create small routines and provide solutions to application problems where conceptual knowledge of Linear Algebra is required, thus managing to incorporate them in a transversal way to programming skills that this academic programme promotes as considered by (Bravo, Cedeño, Coello, Coello, & Guerrero, 2019).

This proposal seeks to close the gap between the way current students learn and the traditionally implemented methodology, highlighting the importance of the implementation of technological tools in the study of linear algebra and how they contribute to an interactive didactic between teachers and students, obtaining results such as: low retention in the course which is evident in the results obtained, better use of teacher-student meetings, as well as greater receptivity on the part of the latter towards the subject.
To implement this idea, a guide was created in pdf format containing the topics corresponding to the subject of Linear Algebra with applications focused on the Systems Engineering programme. This guide, as the topics are developed, presents instructions on how to use SageMath by means of concrete examples.
Initially, the software is used to perform numerical calculations and simplify problem solving. As the course progresses, routines are implemented through functions created in Python, under the SageMath environment, which challenge the student to apply their basic programming knowledge and create routines to solve problems in Linear Algebra. Engaging the student in a learning environment characterised by problem analysis, exploration, discovery, conjecture and verification of results.
The designed guide includes workshops to enhance the student's ability to analyse, demonstrate, verify, interpret, conjecture and apply the different concepts of linear algebra.
This paper presents the results of this didactic research, in the teaching and learning of some concepts of Linear Algebra, at the end statistical data are presented that allow us to identify the impact on the academic performance of students when the subject of Linear Algebra is developed with the support of SageMath in the academic periods from 2014 to 2022.

II. METHODOLOGY

For the development of this work, qualitative research is chosen as a method, since, through it, a problem is studied that allows the collection and analysis of direct data of reality in this case of the solutions proposed by the students through the Sage-Math software, and activities that allow to observe more frequent errors in the resolution of the proposed problems. Then an interpretation is made with the students' notes in the course of six semesters.
According to Belloch, the stages of evolution in the use of technological resources by teachers are: access, adoption, adaptation, appropriation and invention (Belloch, 2012), this work has been developed in an evolutionary way in the first three stages, since it was necessary to train in the use of the mathematical software SageMath, identify activities with this software for the punctual support in the subject of Linear Algebra aimed at systems engineering students and finally the technology has been integrated with the previous knowledge of programming that students have for the teaching-learning process of this subject.
In methodological terms, the students are guided by the teacher in the study of topics related to the solution of systems of linear equations and proposed problems that require the creation of functions in the Python programming language under the SageMath environment to obtain possible solutions. For this study, a problem is identified as an activity that requires different complex cognitive processes to work together to find the mechanism that allows them to be solved, mediated by the technological-computer resource: SageMath software that enhances students' ability to discern the correct information and generate the environment to solve the problem, which cannot be solved automatically and requires relating diverse knowledge, on the other hand they do not necessarily have a single solution path, allowing them to generate and modify knowledge while developing skills and abilities by practising what they have learnt, awakening their interest and motivation.
This work seeks to provide challenges and strategies in the teaching of Linear Algebra, which, as in (Andrews, Berman,

Stewart, & Zandieh, 2018), illuminate particular theoretical developments on the learning of this science and promise to engage students to promote their understanding.

The didactic proposal that has been implemented in this study includes activities formulated based on the texts: (Grossman, 2008), (Martinez & Sanabria, 2014), (Kolman & Hill, 2006), some of which are presented below. It should be noted that the student can work from their mobile phone, pc or tablet, with the SageMath software online or desktop.

The general objective of these activities is to stimulate learning the solution of systems of linear equations, their geometric interpretation in cases where the number of unknowns is less than or equal to 3, in applications to problems in different areas mediated by SageMath and identifying behaviours of different mathematical phenomena to analyse, classify, deduce, conjecture, characteristics and properties.

III. ACTIVITIES

A. *Activity 1. Lines and Planes in $\mathbb{R}^2$ and $\mathbb{R}^3$*

1) *Aim of the activity:*

- To develop students' analytical and abstract thinking skills by asking them for a strategy to solve a situation using SageMath.
- Motivate students by giving them the possibility of implementing the knowledge of the Systems Engineering programme.
- Stimulate the intuitive part of the concept of a straight line in $\mathbb{R}^2$ and $\mathbb{R}^3$.

i. Define a function in SageMath to find the equation of the line in $\mathbb{R}^2$ and another one to print its graph, given two points in the plane and using determinants theory. Finally, use an example to run the function.

Solution:

```
var('y')
def ecuacionrectarr(v1,v2):
    M=Matrix([[x,y,1],[v1[0],v1[1],1],[v2[0],v2[1],1]])
    EC=M.det()==0
    return show(EC)
```

```
u=vector([2,2])
v=vector([-1,0])
ecuacionrectarr(u,v)
```

$2x - 3y + 2 = 0$

```
var('y')
def graecionrectarr(v1,v2):
    M=Matrix([[x,y,1],[v1[0],v1[1],1],[v2[0],v2[1],1]])
    EC=M.det()==0
    Gra=implicit_plot(EC,[x,-8,8],[y,-8,8],color='black',figsize=
[4,4],ymin=-6,ymax=6,frame=true,gridlines=True,axes_labels=
['$x$','$y$'],axes_labels_size=1.2)
    return show(Gra)
```

```
u=vector([2,2])
v=vector([-1,0])
graecionrectarr(u,v)
```

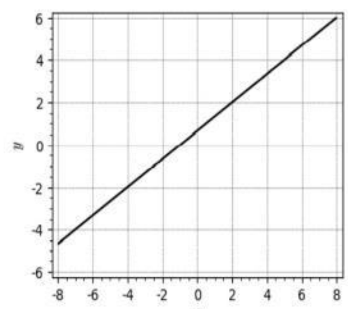

Fig. 1. Geometry of the line in $\mathbb{R}^2$. Source: self-made.

ii. Create a function in SageMath that calculates the parametric equation of the line in R3 and another that prints its graph, if two points through which it passes are known. Then call the function to find the equation of the line passing through the points $P_0(2,3,-4)$ and $P_1(3,-2,5)$.

Solution:

```
def ecurectarrrdospuntos(v1,v2):
    X=v1[0]+(v2-v1)[0]*t
    Y=v1[1]+(v2-v1)[1]*t
    Z=v1[2]+(v2-v1)[2]*t
    return show("x =\t ", X),show("y = \t ", Y),show("z = \t ", Z)
```

```
p1=vector([2,3,-4])
p2=vector([3,-2,5])
ecurectarrrdospuntos(p1,p2)
```

$x = t + 2$

$y = -5t + 3$

$z = 9t - 4$

Fig. 2. Equation of the line in $\mathbb{R}^3$. Source: self-made

```
var('y,z,t')
def grecurectarrrdospuntos(v1,v2):
    X=v1[0]+(v2-v1)[0]*t
    Y=v1[1]+(v2-v1)[1]*t
    Z=v1[2]+(v2-v1)[2]*t
    fx=parametric_plot3d((X,Y,Z),(t,-4, 4),color='black',figsize=
[4,4],ymin=-6,ymax=6,frame=true,gridlines=True,axes_labels=
    ['$x$','$y$'],axes_labels_size=1.2)
    return show(fx)
```

```
p1=vector([2,3,-4])
p2=vector([3,-2,5])
grecurectarrrdospuntos(p1,p2)
```

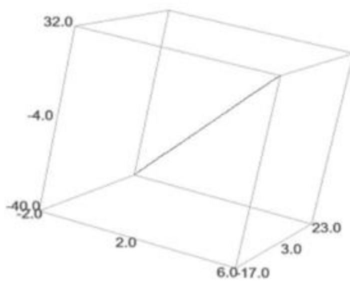

Fig. 3. Geometry of the line in $\mathbb{R}^2$. Source: self-made

iii. Create a function using SageMath to find the equation of the plane containing three non-collinear points. Use this function to determine the equation of the plane passing through the points $(2,-1,2)$, $(-1,0,3)$ and $(4,-3,4)$.

Solution:

```
#ecuación del plano
var('x,y,z')
def ecuacionplano(p1,p2,p3):
    v1=p2-p1
    v2=p3-p1
    n=v1.cross_product(v2)
    ec=(vector([x,y,z])-p1).dot_product(n)==0
    return show(ec)
```

```
p1=vector([2,-2,1])
p2=vector([-1,0,3])
p3=vector([4,-3,4])
ecuacionplano(p1,p2,p3)
```

$8x + 13y - z + 11 = 0$

Fig. 4. Plane equation in $\mathbb{R}^3$. Source: self-made.

B. Conclusions of the activity:

Routines were obtained where it is necessary to identify the components that define the equation of a straight line and a plane. This allows a deep understanding and appropriation of these concepts.

C. Activity 2 Ortogonal Proyection

2) Aim of the activity:

- To stimulate the intuitive idea of the concept of orthogonal projection in the student.
- To develop the student's level of deduction in order to order and direct their ideas.
- Encourage the use of the student's basic knowledge of programming to visualise the orthogonal projection.

i. If $\bar{u} \neq 0$ and $\bar{v} \in \mathbb{R}^n$, we define $proy_{\bar{u}}\,\bar{v}$ the orthogonal projection of the vector $\bar{v}$ onto u, as the vector

$$proy_{\bar{u}}\ \bar{v} = \left(\frac{\bar{u}\bar{v}}{\|\bar{u}\|^2}\right)\bar{u}$$

a) Define a function in SageMath that calculates the projection of one vector onto another and prints the graph of $\bar{u}, \bar{v}$ and $proy_{\bar{u}}\ \bar{v}$. Show with an example its use.

b) Use the function created above on two fixed vectors $\bar{u}$, $\bar{v}$ of $\mathbb{R}^3$ and calculate the projection u, v. Show with an example its use.

c) By calling $v_c = \bar{v} - proy_{\bar{u}}$ what conclusions can we infer if

I. $\bar{u}$ and $\bar{v}$ are orthogonal
II. $\bar{u}$ y $\bar{v}$ and v are parallel
III. Create a function that reports by means of a message what happens in each case and prints the graphs.

Solution a:

```
#proyección ortogonal de v1 sobre v2
def proyort(v1,v2):
    p=v1.dot_product(v2)
    nor=v2.norm()
    proyec=(p/nor^2)*v2
    return proyec
```

Fig. 5. Orthogonal projection. Source: self-made.

Solution b.

```
u=vector([1,-1,-1])
v=vector([1,-1,1])
proyort(v,u)
show(proyort(v,u))
```

$\left(\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$

Fig. 6. Orthogonal projection calculation. Source: self-made.

Solution 1c: To give this solution the student relied on the dot product and cross product to determine whether the vectors were parallel or orthogonal.

```
def proyecc(v,u):
  p=v.dot_product(u)
  n1=v.norm()
  n2=u.norm()
  ppunto=v.dot_product(u)
  norma=u.norm()
  pry=(ppunto/norma^2)*u
  vc=v-pry
  if (ppunto==0):
      show("Los vectores son ortogonales y vc= " , v)
  if (v.cross_product(u)==0):
      show("Los vectores son paralelos y vc= " , (0,0, 0))

  return plot(pry,color='red')+plot(u,color='purple')+plot(v)+plot(vc,color='green')
```

Fig. 7. Function orthogonal projection. Source: self-made.

```
v=vector([sqrt(3),2,1])
u=vector([sqrt(3),-1,-1])
u.dot_product(v)
proyec(v,u)
```

Los vectores son ortogonales y vc= $(\sqrt{3}, 2, 1)$

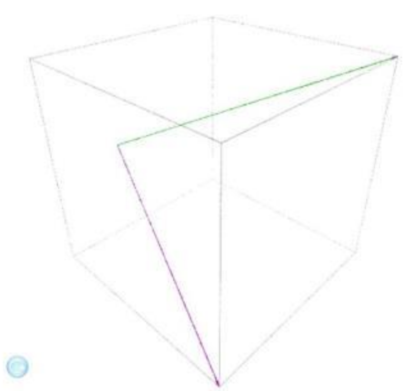

Fig. 8. Geometry of the orthogonal projection when the vectors are orthogonal using Fig. 7. Source: self-made.

```
v=vector([1,2,4])
u=vector([3,6,12])
u.dot_product(v)
proyec(v,u)
```

evaluate

Los vectores son paralelos y vc= $(0, 0, 0)$

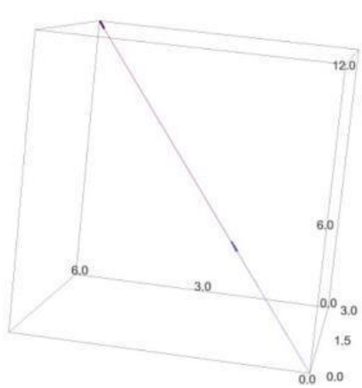

Fig. 9. Geometry of the orthogonal projection when the vectors are parallel using Fig. 7. Source: self-made.

Solution 2c: In this solution the student calculates the angle between two vectors to determine whether the vectors are parallel or orthogonal.

```
def proyec(v,u):
    p=v.dot_product(u)
    n1=v.norm()
    n2=u.norm()
    ang=(arccos(p/(n1*n2))*180)/pi
    ppunto=v.dot_product(u)
    norma=u.norm()
    pry=(ppunto/norma^2)*u
    vc=v-pry
    if (ang==90):
       show("El angulo entre los vectores es : \t" , ang,  "\t luego los vectores son ortogonales")
    if (ang==180 or ang==0):
       show("El angulo entre los vectores es :\t" , ang,  "\t luego los vectores son paralelos")
    return plot(pry,color='red')+plot(vc,color='green')+plot(u,color='purple')+plot(v)
```

Fig. 10. Function orthogonal projection using dot product and cross product. Source: self-made.

```
v=vector([1,2,4])
u=vector([3,6,12])
u.dot_product(v)
proyec(v,u)
```

evaluate

El angulo entre los vectores es : 0 luego los vectores son paralelos

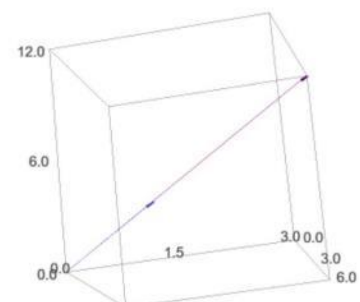

Fig. 11. Geometry of the orthogonal projection when the vectors are orthogonal using Fig. 10. Source: self-made.

```
v=vector([sqrt(3),2,1])
u=vector([sqrt(3),-1,-1])
u.dot_product(v)
proyec(v,u)
```

El angulo entre los vectores es : 90 luego los vectores son ortogonales

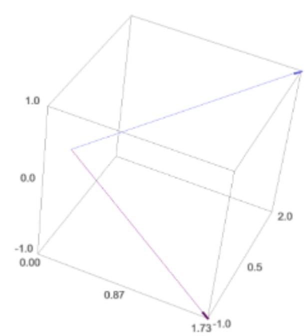

Fig. 12. Geometry of the orthogonal projection when the vectors are parallel using Fig. 10. Source: self-made.

D. Conclusions of the activity:

The power of visualisation that SageMath allows makes the student obtain an intuitive idea of the orthogonal projection of one vector on another and establish criteria between u and v that allow him to infer relationships between the projection vectors v and vc. It is noteworthy that having different solutions enriches the feedback experience, exploring the different results of Linear Algebra.

E. *Activity 3. Solving systems of linear equations*

3) Aim of the activity:

- To guide the student to explore and implement the results of determinant, inverse matrix and the application these have to the solution of non-homogeneous systems of linear equations.
- Motivate students to develop ideas from their area of study to provide solutions to problems that are modelled by means of systems of linear equations.

i. Create a function in SageMath that receives as parameters the matrix of coefficients and the vector of independent terms of a system of non-homogeneous linear equations with number of equations equal to the variables, the function should be as efficient as possible to report:

- ✓ If the system has only one solution, it must output a message and calculate the solution.
- ✓ In case the system has infinite solutions, it must give a message informing and calculating the solution set.
- ✓ The function must contemplate the case in which the system has no solution and report it.

Solution 1: In this solution the student uses matrix rank theory to create the function.

```
def solucionSistema(A,b,n):
    Aaux=A.augment(b,subdivide=true)
    Aaux=Aaux.echelon_form()
    rangoA=rank(Aaux)
    rango=rank(A)
    if(rango!=rangoA):
        show("El sistema no tiene solucion")
    if(rango==n and rango==rangoA):
        s=A.solve_right(b)
        show("El sistema tiene una unica solucion, El vector solucion es =",s)
    if(rango<n and rango==rangoA):
        show("El sistema tiene soluciones infintas")
```

Fig. 13. Solution of homogeneous linear systems of equations that have the same number of equations as unknowns using the range of matrices. Source: self-made.

Solution 2: This solution presents an error, it takes advantage of the student's error where he does not take into account what could happen if a system despite having a determinant equal to zero does not have a solution. Here, in the example, although the function is in error, SageMath does indicate that the system has no solution by means of a message. This example is used to confront the student's analysis.

```
def sisnohom(A,b):
    aum=A.augment(b,subdivide=True)
    if (A.det() != 0) :
        print('El sistema tiene única solución')
        sol=A.inverse()*b
        print('La solucion es: ',sol)
    else:
        print ('El sistema no tiene única solución')
```

```
A=matrix([[2,4],[2,4]])
b=vector([3,1])
show("El determinante de A= ",A.det())
sisnohom(A,b)
solve([2*x+4*y==3,2*x+4*y==1],x,y)
```

```
El determinante de A=0

El sistema no tiene única solución
Traceback (click to the left of this block for traceback)
...
ImportError: cannot import name domain
```

Fig. 14. Solution of homogeneous linear systems of equations that have the same number of equations as unknowns, making use of the determinant theory. Source: self-made.

Solution 3: In this case the student uses theoretical aspects of the inverse of a matrix to calculate the solution in the case of having only one solution, but does not analyse the case of infinite solutions, nor does he calculate the solution set and the one with no solution. The execution of the function is shown for a particular system.

```
def sisnohom(A,b):
    aum=A.augment(b,subdivide=True)
    if (A.det != 0) :
        print('El sistema tiene única solución')
        sol=A.inverse()*b
        print('La solucion es: ',sol)
    else:
        print ('El sistema tiene infinitas soluciones')
```

```
A=matrix([[2,2],[2,4]])
b=vector([3,1])
sisnohom(A,b)
```

```
El sistema tiene única solución
('La solucion es: ', (5/2, -1))
```

Fig. 15. Solution of homogeneous linear systems of equations that have the same number of equations as unknowns, using the inverse matrix theory. Source: self-made.

Solution 4: Here we see another solution, the student succeeds in calculating the solution set when the system has a coefficient matrix of size three by three. It does indeed work, but only if the system has at least one solution. The execution of the function to a particular system is shown.

```
def calcularsolucion(M,v):
    k=M.det()
    if (k !=0):
        print("El sistema tiene única solución")
        aumentada=M.augment(v,subdivide=true)
        solucionunica=M.solve_right(v)
        show('Y la solución para x,y,z es:' , solucionunica)
    else:
        print('El sistema tiene infinitas soluciones')
        var('x,y,z')
        show("El conjunto solucion  es:"
,solve([M[0,0]*x+M[0,1]*y+M[0,2]*z==v[0],M[1,0]*x+M[1,1]*y+M[1,2]*z==v[1],M[2,0]*x+M[2,1]*y+M[2
```

```
M=matrix(QQ,3,3,[2,4,6,4,5,6,2,7,12])
v=vector([18,24,30])
calcularsolucion(M,v)
```

El sistema tiene infinitas soluciones

El conjunto solucion es: $[[x = r_3 + 1, y = -2r_3 + 4, z = r_3]]$

Fig. 16. Solution of homogeneous linear systems of equations that have the same number of equations as unknowns, making use of the determinant theory. Find the solution for the case where the system has three unknowns and three equations. Source: self-made.

In case the system has no solution, the function shows a contradiction, which is used in class session for feedback with the following example: the function is called and evaluated on a zero matrix associated to a non-homogeneous system whose determinant is equal to zero and that the system has no solution, the function although calculating the solution set correctly the message "the system has infinite solutions" contradicts that solution.

```
M=matrix(QQ,3,3,[2,4,6,4,5,6,2,7,12])
show('Determinante de M=' , det(M))
v=vector([18,24,36])
calcularsolucion(M,v)
```

Determinante de M=0

El sistema tiene infinitas soluciones

El conjunto solucion es: []

Fig. 17. Counter example. The error that can be presented by the functions created by the students to solve systems of linear equations is indicated. Source: self made.

These solution types are analysed in the classroom in order to detail errors in the solution of the activity by means of feedback and to inform on how to overcome these difficulties by means of a feedforward.

2. Use the function created to solve the following problem and interpret the solution: a state fish and game department provides three types of food to a lake that is home to three species of fish. Each fish of species 1 consumes an average of 1 unit of food 1, 1 unit of food 2 and 2 units of food 3 each week. Each fish of species 2 consumes an average of 3 units of food 1, 4 units of food 2 and 5 units of food 3 each week. For a fish of species 3, the average weekly consumption is 2 units of food 1, 1 unit of food 2 and 5 units of food 3. 25,000 units of food 1, 20,000 units of food 2 and 55,000 units of food 3 are provided to the lake each week. If we assume that the fish eat all the food, how many fish of each species can coexist in the lake?

Using the function created in solution 1 of item 1, we obtain:

```
A=matrix(QQ,3,3,[1,3,2,1,4,1,2,5,5])
A1=A.augment(b,subdivide=true)
show(A1)
show(A1.echelon_form())
b=vector([25000,20000,55000])
solucionSistema(A,b,3)
```

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 25000 \\ 1 & 4 & 1 & 20000 \\ 2 & 5 & 5 & 55000 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 40000 \\ 0 & 1 & -1 & -5000 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

El sistema tiene soluciones infintas

$x + 5z = 40000 \quad x = 40000 - 5z$

$y - z = -5000 \quad y = z - 5000$

Fig. 18. Application of systems of linear equations. Source: self-made

.

For fish to coexist it is concluded that 5000<z<8000, since the number of fish of each species must be strictly greater than zero.

F. *Conclusions of the activity:*

There are students who manage to interpret according to the data of the problem the solution that the software yields by means of the created function. Others do not realise the domain of the free variable, which in this case is z, so that the fish could coexist.

G. *Activity 4. Geometric interpretation of the solution of systems of linear equations in $\mathbb{R}^3$.*

1) *Aim of the activity:*

- To implement SageMath to graph the geometric objects corresponding to the equations of a system of linear equations with three unknowns and its solution, so that from observation the student obtains an intuitive idea of the type of solution set.
- Establish a relationship between the solution set of systems of linear equations and the intercept of planes in space.
- Coordinate the algebraic and graphic representation registers in the process of solving systems of linear equations.

i. Consider the following system of linear equations.

$$\begin{aligned} x + 2y - z \quad &= 0 \\ x + 2y - z + 4 \quad &= 0 \end{aligned}$$

Use SageMath to graph the equations of the system and to infer the type of solution set

Solution:

```
var('x,y,z,t')
ec1=x+2*y-z==0
ec2=x+2*y-z==4
implicit_plot3d(ec1,(x,-10,10),(y,-10,10),(z,-10,10),color='red') + implicit_plot3d(ec2,(x,-10,10),(y,-10,10),
(z,-10,10),color='blue')
```

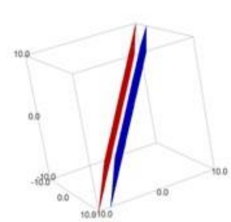

Fig. 19. Geometry of the solution set type of a system of linear equations with two equations and three unknowns, when the solution set is empty. Source: self-made.

ii. Consider the following system of linear equations.

$$2x + 3y - 4z = 2$$
$$4x + 6y - 8z = 4$$
$$6x + 9y - 12z = 6$$

Use SageMath to graph the equations of the system and infer the type of solution set.

Solution:

```
var('x,y,z,t')
ec1=2*x+3*y-4*z==2
ec2=4*x+6*y-8*z==4
ec3=6*x+9*y-12*z==6
implicit_plot3d(ec1,(x,-10,10),(y,-10,10),(z,-10,10),color='red') + implicit_plot3d(ec2,(x,-10,10),(y,-10,10),
(z,-10,10),color='blue')+ implicit_plot3d(ec3,(x,-10,10),(y,-10,10),(z,-10,10),color='yellow')
```

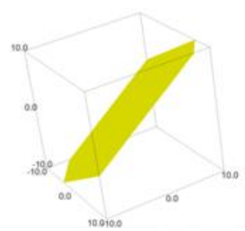

Fig. 20. Geometry of the solution set type of a system of linear equations with three equations and three unknowns, when the solution set is infinite. Source: self-made.

iii. Consider the following system of linear equations.

$$2x + 3y - 4z + 5 = 0$$
$$3x + 2y + 5z + 6 = 0$$

Solve the system in SageMath. Interpret geometrically each equation and the solution set of the system.

```
var('x,y,z,t')
ec1=2*x+3*y-4*z+5==0
ec2=-3*x+2*y+5*z+6==0
sol=solve([ec1,ec2],x,y,z)
show(sol)
```

$$\left[\left[x = \frac{23}{13}r_4 + \frac{8}{13}, y = \frac{2}{13}r_4 - \frac{27}{13}, z = r_4\right]\right]$$

```
implicit_plot3d(ec1,(x,-4,4),(y,-4,4),(z,-4,4),color='red') + implicit_plot3d(ec2,(x,-4,4),
(y,-4,4),(z,-4,4),color='green',size=4)+parametric_plot3d(((23/13)*t+(8/13),(2/13)*t-
(27/13),t),(t,-4,4),color='yellow')
```

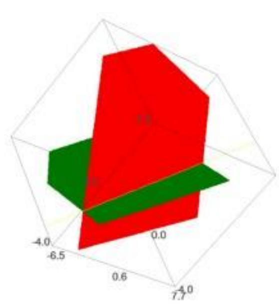

Fig. 21. Equation and geometry of the solution of systems of linear equations with two equations and three unknowns, when the solution set is infinite. Source: self-made.

iv. Give three systems of three-by-three equations with different solution sets and relate the type of solution set to the graph.

Solution:
Three planes whose intersection is a point.

```
var('x,y,z')
ec1=2*x+3*y-4*z==2
ec2=x-5*y+z==1
ec3=7*y-z==-3
sis1=[ec1,ec2,ec3]
show(solve(sis1,x,y,z))
implicit_plot3d(ec1,(x,-4,4),(y,-4,4),(z,-4,4),color='red') + implicit_plot3d(ec2,(x,-4,4),(y,-4,4),
(z,-4,4),color='green')+implicit_plot3d(ec3,(x,-4,4),(y,-4,4),(z,-4,4),color='blue')
```

$$\left[\left[x = \left(-\frac{22}{29}\right), y = \left(-\frac{18}{29}\right), z = \left(-\frac{39}{29}\right)\right]\right]$$

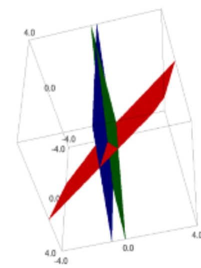

Fig. 22. Equation and geometry of the solution of systems of linear equations with three equations and three unknowns, when the solution set is a point. Source: self-made.

Three parallel planes

```
ec1=2*x+3*y-4*z==5
ec2=4*x+6*y-8*z==4
ec3=6*x+9*y-12*z==-4
sis3=[ec1,ec2,ec3]
show(solve(sis3,x,y,z))
implicit_plot3d(ec1,(x,-4,4),(y,-4,4),(z,-4,4),color='red') + implicit_plot3d(ec2,(x,-4,4),(y,-4,4),
(z,-4,4),color='green')+implicit_plot3d(ec3,(x,-4,4),(y,-4,4),(z,-4,4),color='blue')
```

[]

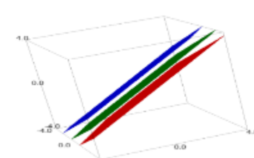

Fig. 23. Equation and geometry of the solution of systems of linear equations with three equations and three unknowns, when the solution set is empty. Source: self-made.

Three planes that do not intersect at the same time

```
var('x,y,z')
ec1=x-y+z==0
ec2=-x+3*y+z==5
ec3=3*x+y+7*z==11
sis1=[ec1,ec2,ec3]
show(solve(sis1,x,y,z))
implicit_plot3d(ec1,(x,-3,3),(y,-2,2),(z,-2,2),color='red') + implicit_plot3d(ec2,(x,-3,3),(y,-2,2),
(z,-2,2),color='green')+implicit_plot3d(ec3,(x,-3,3),(y,-2,2),(z,-2,2),color='blue')
```

[]

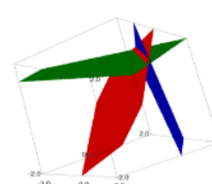

Fig. 24. Equation and geometry of the solution of systems of linear equations with three equations and three unknowns, when the solution set is empty. Source: self-made.

Three planes that do not intersect at the same time.

```
var('x,y,z,t')
ec1=x+y-2*z==9
ec2=2*x-y+4*z==4
ec3=3*x+2*z==13
sis1=[ec1,ec2,ec3]
show(solve(sis1,x,y,z))
implicit_plot3d(ec1,(x,-20,20),(y,-20,20),(z,-20,20),color='red') + implicit_plot3d(ec2,(x,-20,20),(y,-20,20),
(z,-20,20),color='green')+implicit_plot3d(ec3,(x,-20,20),(y,-20,20),(z,-20,20),color='blue')+parametric_plot3d(((-2/3)*t+(13/3),
(8/3)*t+(14/3), t),(t,-20,20),color='yellow')
```

$$\left[\left[x = -\frac{2}{3}r_1 + \frac{13}{3}, y = \frac{8}{3}r_1 + \frac{14}{3}, z = r_1\right]\right]$$

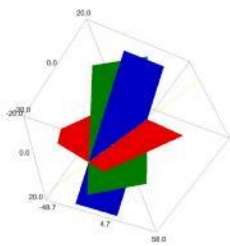

Fig. 25. Equation and geometry of the solution of systems of linear equations with three equations and three unknowns, when the solution set is infinite. Source: self-made.

H. *Conclusions of the activity:*

The visualisation allowed by SageMath makes it possible for the student to obtain a geometric interpretation of the type of the solution set of the system. By correlating the graphical representation of the solution set with its algebraic representation, the student can identify the need to modify the domains of the variables and thus fully understand the meaning of the solution set.

I. *Activity 5. Geometric interpretation of linear independence/dependence between vectors of* $\mathbb{R}^3$.

1) *Aim of the activity:*

- To study the different tools of SageMath together with the theoretical results of Linear Algebra for the study of linear dependence and independence between vectors of the usual vector space $\mathbb{R}^3$.

Consider the following definition and solve:

i. Let $\{v_1, v_2, v_3, \ldots, v_n\}$ be a set of vectors in a vector space V. These vectors are linearly dependent (LD) if there exist n scalars $c_1, c_2, c_3, \ldots, c_n$ not all null such that:

$$c_1v_1 + c_2v_2 + c_3v_3 + \ldots + c_nv_n = 0.$$

If the vectors are not linearly dependent they are said to be linearly independent (LI).

a. Determine whether the set of vectors $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\-2\\3\end{bmatrix}\right\}$ are LI or LD, in the usual vector space $\mathbb{R}^3$.

Solution 1: In this solution the vectors are written as columns of a matrix that is associated to a homogeneous system, the matrix is scaled, the student deduces from the scaled form of the matrix whether the vectors are LD or LI, using the definition.

```
A=matrix(QQ,[[1, 0, 0], [0, 1, 1],[-1, -2, 3],])
E=A.echelon_form()
show("Matriz Aumentada = ", A)
show("Matrix escalonada = ", E)
```

$$\text{Matriz Aumentada} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & -2 & 3 \end{pmatrix}$$

$$\text{Matrix escalonada} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Fig. 26. Analysis of linear dependence/independence of vectors in $\mathbb{R}^3$. Using the coefficient matrix scaling of the associated homogeneous system resulting from posing the solution to the problem. Source: self-made.

Solution 2: As the matrix is square, the student uses the functions created in activity three, given that to answer this question it is necessary to know the solution set of the system of homogeneous linear equations since we are considering the trivial zero in the usual vector space $\mathbb{R}^3$. The student uses the fact that if the system has only one solution it must be the trivial one, and in this case the vectors are LI, if the system has solutions other than the trivial one the vectors are LD.

```
M=matrix(QQ,3,3,[1,0,0,0,1,1,-1,-2,3])
show('Determinante de M=' , det(M))
v=vector([0,0,0])
calcularsolucion(M,v)
```

evaluate

Determinante de M=5

El sistema tiene única solución

Y la solución para x,y,z es: (0, 0, 0)

Fig. 27. Analysis of linear dependence/independence of vectors in $\mathbb{R}^3$. Using determinant theory applied to the coefficient matrix of the associated homogeneous system resulting from posing the solution to the problem. Source: self-made

Another way in which SageMath can help answer whether a set of vectors is LI or LD, in a more simplistic way which does not allow the student to explore the necessary theory but facilitates the creation of functions to make the graphs is: with the command “are_linearly_dependent(vecs)” For example:

```
V = QQ^3
vecs = [V([1, 2, -1]), V([3, 2, 5]), V([1, -2, 3])]
V.are_linearly_dependent(vecs)
    False
V = QQ^3
vecs = [V([1, 1, 0]), V([2, -2, 3]), V([6, -6, 9])]
V.are_linearly_dependent(vecs)
    True
```

Fig. 28. Analysis of linear dependence/independence of vectors in $\mathbb{R}^3$. Using SageMath's own functions. Source: self-made.

b. Create a function that receives in its argument three vectors of the usual vector space R3 and returns if the set of these three vectors are LI or LD and the graph. Illustrate its use with examples.

Solution:

We see in the student's solution the use of SageMath's own commands for the creation of a new function, which in this case indicates whether the vectors are LI or LD, but in addition returns the graph of the vectors.

Example 1:

```
def veclild(v1,v2,v3):
    V = QQ^3
    vecs= [V1, V2, V3]
    valor=V.are_linearly_dependent(vecs)
    if valor==False:
        print('Los vectores son linealmente independientes')
    else:
        print('Los vectores son linealmente dependientes')
    return  plot(v1,color='green')+plot(v2,color='red')+plot(v3)
```

```
v1=vector([1, 2, -1])
v2=vector([3, 2,5])
v3=vector([1, -2, -3])
veclild(v1,v2,v3)
evaluate
    Los vectores son linealmente independientes
```

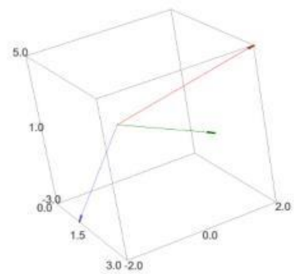

Fig. 29. Geometry of the linear independence of vectors in $\mathbb{R}^3$. Source: self-made.

Example 2:

```
v1=vector([1,2,0])
v2=vector([0,1,0])
v3=vector([2,7,0])
veclild(v1,v2,v3)
    Los vectores son li
```

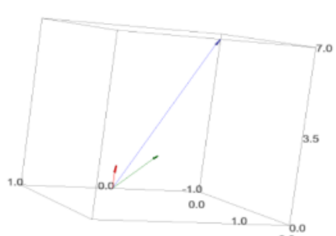

Fig. 30. Geometry of the linear dependence of vectors in $\mathbb{R}^3$. Source: self-made.

Example 3:

```
v1=vector([1, 1, 0])
v2=vector([2, -2,3])
v3=vector([6, -6, 9])
veclild(v1,v2,v3)
evaluate
    Los vectores son linealmente dependientes
```

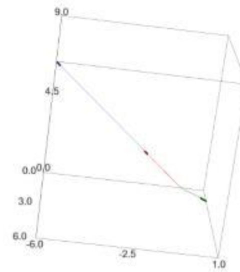

Fig. 31. Geometry of the linear dependence of vectors in $\mathbb{R}^3$. Three vectors in the same plane. Source: self-made.

Conclusions of the activity:

Item 1 of this activity makes it possible to relate the definition of linearly dependent and independent set to the solution of a system of linear equations its associated scalar matrix, on the contrary, item 2 provides an automatic solution of this definition, but with a visual potential that allows to identify that a set of three vectors in the usual vector space $\mathbb{R}^3$ are linearly independent if they lie in different planes.

IV. RESULT

An analysis of the courses AL-2014 group D, AL-2014 group K, AL-2016 group J, AL-2018 group J, AL-2018-2, AL-2019 group A, AL-2019 group Q, AL-2019 group J, AL-2020 group-I, AL-2021 group F and AL-2021 group Q, of the subject Linear Algebra with students of the systems engineering programme of the Universidad del Cauca is presented below, using frequency histograms and infograms that allow inferring the influence of the use of the SageMath software in the retention of these courses. The histograms that show the frequency of the final grades obtained by the students in each Linear Algebra course in the different years and academic periods mentioned are described below.

The following histograms are the one corresponding to the course AL-2014 group D. In this course 75% of the students had grades less than or equal to 2.85 and only 26% passed and other histogram of the course AL-2014, group K (Linear Algebra group K of the year 2014). In this course 50% of the students had final grades less than or equal to 2.5 and 49% passed the course.

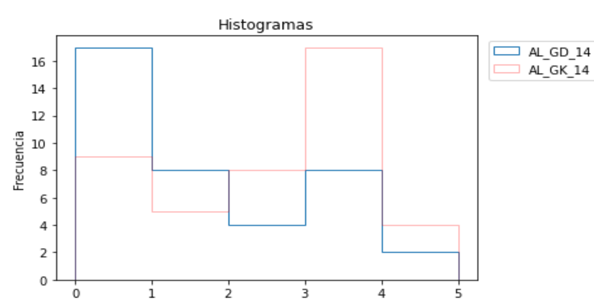

Fig. 32. Final grades corresponding to group J and group K of Linear Algebra in the year 2014 respectively. Source: self-made.

We continue with the histograms corresponding to the final grades of the courses AL-2016 and the course AL-2018 group J, in the first case 50% of the students have grades below 2.5 and only 26% pass the course and in the second case the 50% of the students had final grades less than or equal to 2.2 and only 38% pass the course.

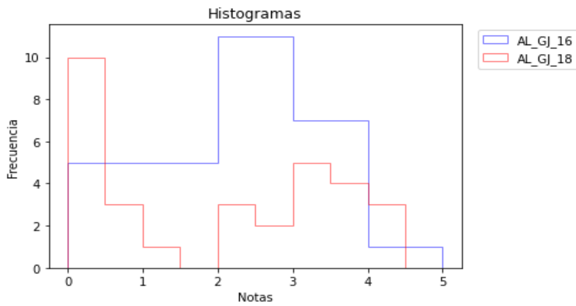

Fig. 33. Final grades corresponding to groups J of Linear Algebra in the years 2016 and 2018. Source: self-made.

For the year 2018 in the second period, the SageMath software is incorporated with the methodology presented in this work, in the course AL-2018, 95% of the students passed the course, beginning to show a low retention in the subject.

Likewise, in the AL-2019 group A course, 67% of the students pass the course, a slightly lower percentage, but still a good percentage of students who pass compared to the courses in which SageMath was not used.

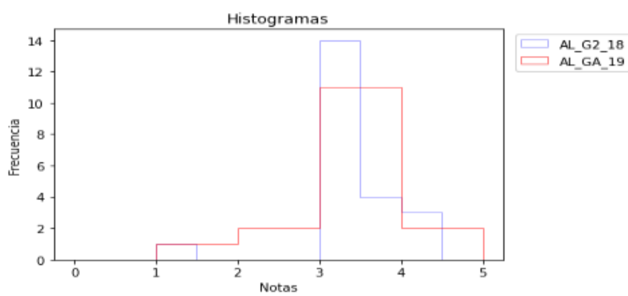

Fig. 34. Final grades corresponding to groups 2 and group A of Linear Algebra in the years 2018 and 2019 respectively. Source: self-made.

In the courses AL-2019 group J 64% of the students passed and group AL-2019 group Q the 81% of the students passed the course.

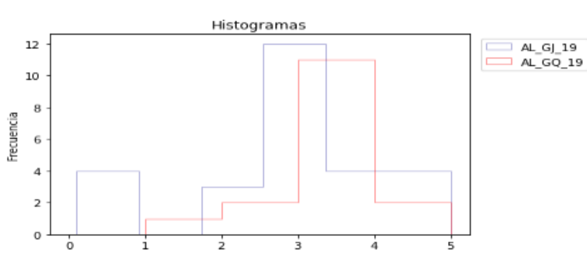

Fig. 35. Final grades corresponding to group J and group Q of Linear Algebra in the year 2019 respectively. Source: self-made.

Finally, in the years 2020 and 2021 it is notherworthy that we were in virtuality due to the pandemic, circumstance that limits the work given the difficulty of access of the students to computer equipment and internet, this caused a premature withdrawal by the students.

Although there was a decrease in the number of students who passed the subject Al-2020 group I, it is still a good result. Here 50% of the students passed. In the AL-2021 group F course, we see a high number of students passed the course, to be more precise, 81%. In the course of Al 2021, group Q, the percentage of students who pass the course drops to 50%. It is worth noting that, when implementing an innovative pedagogical strategy, not all courses respond in the same way, but even so, the results obtained show a drop in retention.

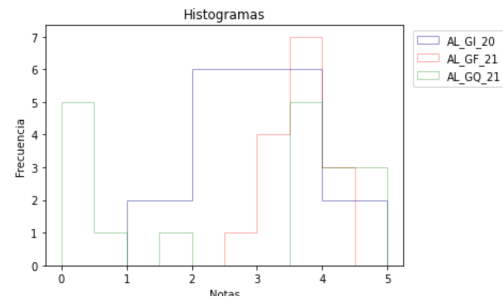

Fig. 36. Final grades corresponding to group I 2020, group F and Q of Linear respectively Algebra in the year 2021. Source: self-made.

To illustrate the impact on student retention in this subject, the following graph shows the courses in their respective periods versus the percentage of students who passed. In the period between 2014 and 2016, we recall, the courses were developed through lectures, tutorials, written mid-term exams, while between 2018 and 2021 the subject is developed with the support of SageMath, and the methodology already explained.

Thus, the percentage of students who pass the subject before incorporating this methodological strategy and after implementing it is compared.

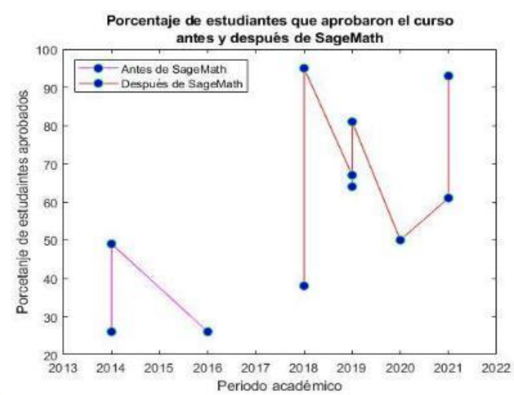

Fig. 37. Each point indicates the percentage of students passing before incorporating SageMath with activities

V. CONCLUSIONS

It is well known that the assimilation and integration of the concepts inherent to Linear Algebra represent a challenge for students given the level of abstraction of some of the topics studied in this subject. This proposal undoubtedly shows that the support of technological tools intercepted with a methodological strategy makes a great difference in the understanding and appropriation of the concepts that is reflected in the academic results according to the final grades of each student.

Innovation in the educational field focused on the use of ICT as a tool for training purposes TAC, allows particularly in the study of Linear Algebra to contribute to improving the training of engineers for the future, responding to the need for curricular changes and teaching-learning processes that reduce desertion in this subject, ensure the quality of learning and contribute to social cohesion with the same interests and common goals among the student population and teaching staff.

Thanks to the reflection of the teaching work and the interaction with the student in the development of the chairs, these innovative ideas arise that leave a reference and invite other teachers to explore new methodologies that represent a change in the way of teaching and learning mathematical concepts. In this spirit, we intend to continue integrating and combining various technologies in the exploration of knowledge and analysis of algebraic objects.

ACKNOWLEDGMENTS

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