Python programming and algebra: Some special elements in Gaussian integers modulo a prime

Programación en Python y álgebra: Algunos elementos especiales en los enteros gaussianos módulo un primo

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***Abstract* — We consider the complexification of a commutative ring with unity and specialize this construction to $\mathbb{Z}_p[i]$, with p a prime of the form $4k+1$. Since this ring is commutative with unity and is not a field, it is feasible to study various classes of special elements such as invertibles, zero-divisors, idempotents, and nilpotents. The method for this study consists of developing computer programs in Python, through which the lists of special elements in $\mathbb{Z}_p[i]$ are generated for different values of p. The patterns that characterize these lists are sought, in addition to the cardinality of each of these sets. Subsequently, conjectures of mathematical type are stated for each of these classes of elements, which reflect the observed patterns and properties. Finally, formal mathematical proofs of all the conjectures found are made based on various concepts and results of the theory of numbers, groups, and rings. Thus, we show that Python programming, properly used as part of a method, becomes an important tool to identify patterns, properties, and characteristics of several abstract concepts, typical of algebra.**

***Index Terms*— Idempotent, invertible, nilpotent, Python, zero-divisor.**

***Resumen* — En este trabajo presentamos la complejificación de un anillo conmutativo con unidad y especializamos esta construcción al anillo de los enteros gaussianos $\mathbb{Z}_p[i]$, con p un primo de la forma $4k+1$. Como este anillo es conmutativo con unidad y no es un cuerpo, resulta viable estudiar diversas clases de elementos especiales como invertibles, divisores de cero, idempotentes y nilpotentes. La metodología seguida para este estudio consiste en desarrollar programas computacionales en Python, mediante los cuales se generan las listas de elementos especiales en $\mathbb{Z}_p[i]$ para distintos valores de p; luego con estas listas se buscan los patrones que caracterizan a los elementos invertibles, idempotentes, divisores de cero y 2-nilpotentes, además del cardinal de cada uno de estos conjuntos. Posteriormente, para cada una de las clases de elementos anteriores se enuncian conjeturas de tipo matemático, las cuales reflejan los patrones y propiedades observadas. Finalmente, apoyados en diversos conceptos y resultados de la teoría de números, grupos y anillos, se hacen las demostraciones matemáticas formales de todas las conjeturas halladas. Mostramos así que la programación en Python, usada adecuadamente en la metodología, se convierte en una herramienta importante para identificar patrones, propiedades y características de diversos conceptos abstractos, propios del álgebra.**

***Palabras clave*— Divisor de cero, 2-nilpotente, idempotente, invertible, Python.**

I. Introduction

COMPLEX numbers, since their appearance in the 16th century due to Girolamo Cardano [1], have served as inspiration for many studies in mathematics, engineering, and physics, including countless applications in control theory, electromagnetism, fluid dynamics, signal processing, quantum mechanics, cosmology and cartography, among many others [2,3]. In some cases, it has been necessary and fundamental to consider subsets of the complex numbers, such as the Gaussian integers or Eisenstein integers, since they are the appropriate environment to study various problems in number theory, such as the laws of quadratic and cubic reciprocity [4], or certain subfields of the complex numbers, important for studying the roots of polynomials, extensions of fields and Galois correspondences [5,6]. Currently, the formal construction of the field of complex numbers is based on taking the complete set of ordered pairs with real entries and considering as operations the usual component by component sum and a special product, which is the one that characterizes the complex numbers [7]. The importance of this construction lies in the fact that any ring can be taken as a basis, as shown in [8] and developed in detail in [9,10], which means that virtually any commutative ring with unity can be complexified. This not only allows the construction of new rings, extended via the complex numbers, but also provides another way to introduce some special sets of the complex numbers, such as the ring of Gaussian integers, the field of rational complex numbers, and also finite rings such as the Gaussian integers modulo n, which will be studied in this paper.

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On the other hand, the current importance of programming and more broadly of computational thinking lies, among many other things, in the fact that it is a fundamental tool for solving complicated problems, automating tasks, and facilitating multiple aspects of contemporary society, a process that has accelerated due to the emergence and development of artificial intelligence. According to [11],

the concept of computational thinking was implicit in Papert [12], where he spoke of its importance in the framework of his educational proposal known as constructionism, through the work with the robot "turtle" and the programming language LOGO. Papert recognized the importance of an education in which technology should be immersed, so he considered it important to include robotics and programming from an early age, which was not achieved in the following decades. However, according to [11], the work that marks a turning point in the current conception of computational thinking is that developed by Wing [13], where she shows that this concept provides new meanings and dimensions for the human being, with great potential to be developed in educational environments. For [14], computational thinking is a term coined by Wing [13], to describe a set of skills, habits, and comprehensive approaches to solving problems related to programming, which are not only limited to computer use. Thus, in a certain way, we can understand that computational thinking is a set of processes that allow active interaction between a person and a computer, to solve problems of various kinds and make use of patterns, algorithms, models, etc.

Given the above, the Ministries of Education and Ministries or Institutes of Technology in many countries have been creating special initiatives aimed at young people, to foster computational skills, including programming. Even in countries such as England, Spain, the United States, Costa Rica, Ecuador, and Argentina, curricular proposals have been developed to enhance computational and technological skills, including computational thinking and programming [15]. In the Colombian case, the Ministry of Information Technologies and Communications has a Digital Government Policy [16], through which it is developing the Colombia Program and Green Code Strategy initiatives. The purpose of the former is to generate resources and opportunities for teachers' professional development to promote computational thinking in official educational institutions in Colombia, with a focus on gender equity. The second consists of the first learning ecosystem for the development of computational thinking skills for children and young people in public and private schools in Colombia. The ICT Ministry also has the following four proposals for Free Digital Training: Talento Tech, Senatic, Avanza Tech, and Talento GovTech.

As we have seen, programming, and more broadly computational thinking, has become a fundamental tool in practically all areas of knowledge. In mathematics, this tool has been important not only for research but also for education. Currently, several programs for mathematical and statistical use are booming, due to their potential, ease of learning, and because they are open source, allowing any user to use them without any subscription costs for their use. In particular, Python programming has provided countless possibilities in research and teaching, which is why we highlight in this work the use of this language for the study of various algebraic concepts.

Our principal goal in this work is to show the importance of programming to obtain results in algebra. This paper is organized as follows. After the introduction, in Section II, we present the construction of the complexification of any commutative ring with unity. In Section III, we use Python programming to continue the work developed in [9,10]. We study the zero-divisors, idempotents, 2-nilpotents, and invertible elements of the ring $\mathbb{Z}_p[i]$ with p a prime of the form $4k+1$. We want to emphasize the importance of programming to obtain results in algebra. Thus, the implemented methodology consists of developing computer programs in Python language. This leads to obtaining the lists of such elements for different values of the prime p. In the next step, we find the corresponding patterns that characterize these classes of elements, which are presented as conjectures. Finally, we proceed to find the formal proofs of the conjectures obtained, most of which were originally developed by the authors. This same process is done to determine the cardinality of each of these classes.

II. The Complexification Of $\mathbb{Z}_p$, With p Prime

The classical construction of the complex numbers from the real numbers is a topic that is usually addressed in the exercises of an algebra textbook or presented in a course on complex variables. In addition, the three classical representations of the complex numbers, the usual one, the one by matrices, and the one by a quotient, are often overlooked by many students of both engineering and mathematics. The work done in [9,10] recovers these constructions and representations and brings them into more general contexts. Following these two works, we present below the construction of the complexification of any commutative ring with unity.

Let A be a commutative ring with unit element 1. On the set $A \times A$, we define the sum component by component. The product is given by

$$(a,b)(c,d) = (ac - bd, ad + bc),$$

for any $(a,b),(c,d) \in A \times A$.

Affirmation 1. If A is a commutative ring with unit 1, then the set $A \times A$ with the operations given above is a commutative ring with unit.

Proof. Since the sum of pairs is component by component, it is easily observed that $(A \times A, +)$ is an abelian group. Moreover, the product is commutative, since A is commutative. Now for $(a,b),(c,d),(e,f) \in A \times A$ it follows that

$$\begin{aligned}(a,b)\big((c,d)(e,f)\big) &= (a,b)(ce - df, cf + de)\\ &= (ace - adf - bcf - bde,\\ &\quad acf + ade + bce - bdf) \quad (1)\end{aligned}$$

and

$$\begin{aligned}\big((a,b)(c,d)\big)(e,f) &= (ac-bd, ad+bc)(e,f)\\ &= (ace-bde-adf-bcf,\\ &\quad acf-bdf+ade+bce). \quad (2)\end{aligned}$$

Since (1) = (2), then $(a,b)\big((c,d)(e,f)\big) = \big((a,b)(c,d)\big)(e,f)$. That is, the product in $A \times A$ is associative.

Moreover, for $(a,b),(c,d),(e,f) \in A \times A$ it follows that

$$\begin{aligned}(a,b)\big((c,d)+(e,f)\big) &= (a,b)(c+e.d+f)\\ &= (ac+ae-bd-bf,\\ &\quad ad+af+bc+be).\\ &= (a,b)(c,d)+(a,b)(e,f).\end{aligned}$$

That is, the product distributes with respect to the sum on the left. The other distributive property is a consequence of the previous distributivity and commutativity. Finally, it is easy to see that the pair $(1,0)$ is the identity element for the product. Therefore, the set $A \times A$ with the indicated operations is a commutative ring with unity. ■

The ring $(A \times A, +, \cdot)$, described above will be denoted by $A \boxtimes A$. In addition, seeking to make this work complete for the reader, we present below some particularities of this ring.

Affirmation 2. Let A be a commutative ring with unit element 1. Then:

1. The function $\varphi : A \to A \boxtimes A$ defined by $\varphi(a) = (a,0)$ for all $a \in A$ is an injective homomorphism of rings.

2. The element $i = (0,1) \in A \boxtimes A$ commutes with $(m,0)$, for each $m \in A$.

According to Affirmation 2, the First Isomorphism Theorem allows us to conclude that the ring $A \boxtimes A$ contains a subring isomorphic to the ring A or we can also say that any element $a \in A$ is biunivocally identified with the pair $(a,0) \in A \boxtimes A$. Then, for each $(a,b) \in A \boxtimes A$ we have $(a,b) = (a,0) + (b,0)(0,1)$. Moreover, note that the element $i = (0,1)$ satisfies $i^2 = (-1,0)$ and i commutes with each element $(m,0)$. Therefore, using the aforementioned identification we can conclude that $(a,b) = a + bi$ where$a, b \in A$, $i = (0,1)$ is such that $i^2 = -1$ and i commutes with each element$(m,0)$.

Thus, $A \boxtimes A$ can be seen as the "**Complexification of the ring *A***'' [8]. That is, $A \boxtimes A$ coincides with the ring,

$$\mathrm{A}[i] = \{a + bi : a, b \in A, i^2 = -1,\ im = mi, \forall m \in A\}.$$

As particular cases, it is clear that when $A = \mathbb{R}$, the field of complexes $\mathbb{C}$ is obtained; if $A = \mathbb{Z}$, one obtains the ring of Gaussian integers $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ [5,6]; and if $A = \mathbb{Z}_n$, one obtains the Gaussian integers modulo n, $\mathbb{Z}_n[i] = \{a + bi : a, b \in \mathbb{Z}_n\}$, which can also be constructed as the quotient of the ring $\mathbb{Z}[i]$ by the ideal $\langle n \rangle$ in $\mathbb{Z}[i]$ generated by n [17,18].

Since by considering as a base ring the field of reals $\mathbb{R}$, the field of complex numbers $\mathbb{C}$ is obtained, one can generalize this construction by considering finite fields. This is the case in [9,10], where they consider as a base ring the field $\mathbb{Z}_p$, with p a prime, construct the ring $\mathbb{Z}_p[i]$ and show that it is a field if, and only if, p is not a sum of two squares or equivalently p is of the form $4k + 3$. Furthermore, they prove that in this case three isomorphic representations of $\mathbb{Z}_p[i]$, analogous to those obtained from the reals, are obtained. They are obtained using the matrix ring $M(\mathbb{Z}_p) = \left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{Z}_p\right\}$, the quotient ring $\mathbb{Z}_p[x]/\langle x^2 + \overline{1}\rangle$ and the one we have developed in this section $\mathbb{Z}_p \boxtimes \mathbb{Z}_p$ (see Figure 1). Thus, we will denote the ring of Gaussian integers over $\mathbb{Z}_p$ as $\mathbb{Z}_p[i]$ or $\mathbb{Z}_p \boxtimes \mathbb{Z}_p$.

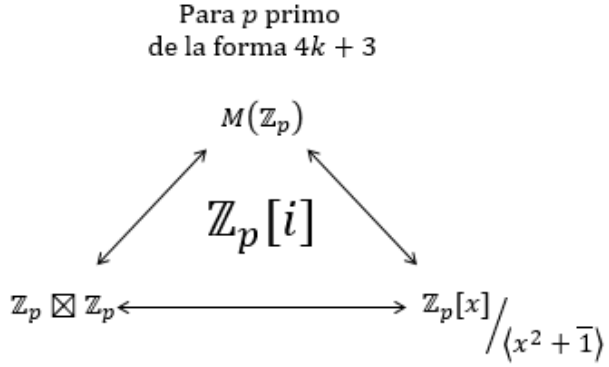

Fig. 1. Isomorphic representations of $\mathbb{Z}_p[i]$ for p prime of the form $4k + 3$.

III. Some Algebraic Concepts In $\mathbb{Z}_p[i]$, With p A Prime Of The Form $4k + 1$

According to the previous section, if p is a prime of the form $4k + 1$, or equivalently p is an odd prime which is the sum of two squares, then $\mathbb{Z}_p[i]$ is a commutative ring with unity, which is not a field. This means that in these rings it is feasible to find various elements important for ring theory such as invertibles, zero-divisors, idempotents, and nilpotents, among many others [5, 8, 19].

Although several of these classes of elements have been characterized in various works [17, 18], the way this has been done is by directly using previous results or theoretical aspects of algebra and number theory to finally, on some occasions, verify these results using programming languages. In this work, we want to emphasize the importance of programming to obtain results in algebra. By continuing the work developed in [9,10], we will use Python to study the zero-divisors, idempotents, 2-nilpotents, and invertible elements of the ring $\mathbb{Z}_p[i]$ with p a prime of the form $4k + 1$. Thus, the process we will follow in this work consists of developing programs in Python and generating the previously described lists of elements of the ring $\mathbb{Z}_p[i]$, for all primes p of the form $4k + 1$ less than 100. Then,

we proceed to identify the patterns or characteristics of each class of these elements, state the conjectures obtained in mathematical terms, and make the corresponding formal proofs. This process is also done to determine the cardinality of each of the sets defined by the different classes of elements. All the results presented here are part of a more general study presented in [20].

Figure 2 shows the program that allows one to perform and visualize the product of elements in $\mathbb{Z}_p[i]$.

```
def simulacionCZp(p):
    lista=list(range(0,p))
    for i in range(0,p):
        for j in range(0,p):
            for k in range(0,p):
                for l in range(0,p):
                    x=[lista[i],lista[j]]
                    y=[lista[k],lista[l]]

                    z=[(x[0]*y[0]-x[1]*y[1])%p,\
                       (x[0]*y[1]+x[1]*y[0])%p]
                    print(x,"*",y,":",z)

    print("multiplicacion de complejos")
```

Fig. 2. Python program for the product in $\mathbb{Z}_p[i]$.

Definition 1. A nonzero element $(\bar{a},\bar{b}) \in \mathbb{Z}_p \boxtimes \mathbb{Z}_p$ is a zero-divisor if there exists a nonzero element $(\bar{c},\bar{d}) \in \mathbb{Z}_p \boxtimes \mathbb{Z}_p$ such that $(\bar{a},\bar{b})(\bar{c},\bar{d}) = (\bar{0},\bar{0})$.

The Python program in Figure 3 was designed to display the zero-divisors for each prime and indicate the number of zero-divisors in each case.

```
def simulacionCZp(p):
    b=[0,0]
    dcero=0
    dcerodist=0
    valores=[]
    lista=list(range(0,p))
    for i in range(0,p):
        for j in range(0,p):
            for k in range(0,p):
                for l in range(0,p):
                    x=[lista[i],lista[j]]
                    y=[lista[k],lista[l]]

                    z=[(x[0]*y[0]-x[1]*y[1])%p,\
                       (x[0]*y[1]+x[1]*y[0])%p]


                    if z==b and x!=b and y!=b:
                        dcero=dcero+1
                        print(x,y)
                        if dcero%(p-1)==1:
                            dcerodist=dcerodist+1


    print("cardinal de la lista:",dcero)
    print("Div. cero distintos:",dcerodist)
```

Fig. 3. Python program for the zero-divisors in $\mathbb{Z}_p[i]$.

The values obtained for $p = 5,\ 13,$ and 17 are shown in Table I. In each pair found by the program, a very interesting particularity is observed. For example, for $p = 5$ if we take the pair of zero-divisors $(\bar{1},\bar{2})$ and $(\bar{1},\bar{3})$, then one has that $\overline{1^2+2^2} = \bar{0}$ and $\overline{1^2+3^2} = \bar{0}$. If for $p = 13$ one takes the pair $(\bar{8},\overline{12})$ and $(\bar{1},\bar{5})$, one has that $\overline{8^2+12^2} = \bar{0}$ and $\overline{1^2+5^2} = \bar{0}$. This pattern is observed for all other pairs of zero-divisors, for all odd primes, sum of two squares less than 100. We can then conjecture that a nonzero element $(\bar{a},\bar{b}) \in \mathbb{Z}_p \boxtimes \mathbb{Z}_p$ is a zero-divisor if, and only if, $\overline{a^2+b^2} = \bar{0}$. Before proving this conjecture, we must prove a preliminary result.

Affirmation 3. Let p be an odd prime which is a sum of two squares. If $(\bar{a},\bar{b})$ is a zero-divisor in $\mathbb{Z}_p \boxtimes \mathbb{Z}_p$, then $\bar{a} \neq \bar{0}$ and $\bar{b} \neq \bar{0}$.

Proof. By assumption, there exists $(\bar{c},\bar{d}) \in \mathbb{Z}_p \boxtimes \mathbb{Z}_p$ nonzero such that $(\bar{a},\bar{b})(\bar{c},\bar{d}) = (\overline{ac-bd},\overline{ad+bc}) = (\bar{0},\bar{0})$. If $\bar{a} = \bar{0}$, then $\overline{-bd} = \bar{0}$ and $\overline{bc} = \bar{0}$. Since $\mathbb{Z}_p$ is an integral domain, one has that $\bar{c} = \bar{0}$ and $\bar{d} = \bar{0}$, which is a contradiction. The same is true if $\bar{b} = \bar{0}$, so we conclude that $\bar{a} \neq \bar{0}$ and $\bar{b} \neq \bar{0}$. ■

TABLE I
ZERO-DIVISORS IN $\mathbb{Z}_5[i]$, $\mathbb{Z}_{13}[i]$, AND $\mathbb{Z}_{17}[i]$

p = 5	p = 13	p = 17
32; distintos 8	288; distintos 24	512; distintos 32
[1, 2] * [1, 3]	[1, 5] * [1, 8]	[1, 4] * [1, 13]
[1, 2] * [2, 1]	[1, 5] * [2, 3]	[1, 4] * [2, 9]
[1, 2] * [3, 4]	[1, 5] * [3, 11]	[1, 4] * [3, 5]
[1, 2] * [4, 2]	[1, 5] * [4, 6]	[1, 4] * [4, 1]
[1, 3] * [1, 2]	[1, 5] * [5, 1]	[1, 4] * [5, 14]
[1, 3] * [2, 4]	[1, 5] * [6, 9]	[1, 4] * [6, 10]
[1, 3] * [3, 1]	[1, 5] * [7, 4]	[1, 4] * [7, 6]
[1, 3] * [4, 3]	[1, 5] * [8, 12]	[1, 4] * [8, 2]
[2, 1] * [1, 2]	[1, 5] * [9, 7]	[1, 4] * [9, 15]
[2, 1] * [2, 4]	[1, 5] * [10, 2]	[1, 4] * [10, 11]
[2, 1] * [3, 1]	[1, 5] * [11, 10]	[1, 4] * [11, 7]
[2, 1] * [4, 3]	[1, 5] * [12, 5]	[1, 4] [12, 3]
[2, 4] * [1, 3]	[1, 8] * [1, 5]	[1, 4] * [13, 16]
[2, 4] * [2, 1]	[1, 8] * [2, 10]	[1, 4] * [14, 12]
[2, 4] * [3, 4]	[1, 8] * [3, 2]	[1, 4] * [15, 8]
[2, 4] * [4, 2]	[1, 8] * [4, 7]	[1, 4] * [16, 4]
[3, 1] * [1, 3]	[1, 8] * [5, 12]	[1, 13] * [1, 4]
[3, 1] * [2, 1]	[1, 8] * [6, 4]	[1, 13] * [2, 8]
[3, 1] * [3, 4]	[1, 8] * [7, 9]	[1, 13] * [3, 12]

Definition 2. Let p be a prime and a an integer relatively prime to p, a relation which is denoted by $(a,p) = 1$. We say that a is a **quadratic residue** or a **square modulo p**, if there exists $b \in \mathbb{Z}$ such that $b^2 \equiv a\ (mod\ p)$.

Note that if a is a quadratic residue modulo p, then there exists $\bar{x} \in \mathbb{Z}_p$ such that $\bar{x}^2 = \bar{a}$. We can then say that $\bar{x}$ is a square root of $\bar{a}$ modulo p and $\overline{-x}$ is also. Then, in this case, we can write $\bar{x} = \pm\sqrt{\bar{a}}$.

For the study of quadratic residues, the Legendre symbol becomes fundamental. It is defined below.

Definition 3. Let p be an odd prime and $a \in \mathbb{Z}$ such that $(a,p) = 1$. Legendre's symbol is defined as:

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & \text{if } a \text{ is a quadratic residue} \\ -1, & \text{if } a \text{ is not a quadratic residue} \end{cases}$$

The following proposition presents some properties of the Legendre symbol. The proofs can be found in many classical texts on number theory [21,22].

Proposition 1. Let p be an odd prime and $a, b \in \mathbb{Z}$ such that $(a,p) = (b,p) = 1$. Then:

1. $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \ (mod\ p)$.
2. $\left(\frac{a^2}{p}\right) = 1$.
3. $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$.
4. If $a \equiv b\ (mod\ p)$, then $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$.
5. $p \equiv 1\ (mod\ 4)$ if, and only if, -1 is a quadratic residue modulo p.
6. $p \equiv 3\ (mod\ 4)$ if, and only if, -1 is not a quadratic residue modulo p.

Affirmation 4. Let p be an odd prime sum of two squares and $(\bar{0},\bar{0}) \neq (\bar{a},\bar{b}) \in \mathbb{Z}_p \boxtimes \mathbb{Z}_p$. Then $(\bar{a},\bar{b})$ is a zero-divisor if, and only if, $\overline{a^2 + b^2} = \bar{0}$.

Proof. If $(\bar{a},\bar{b})$ is a zero-divisor, then there exists $(\bar{c},\bar{d}) \in \mathbb{Z}_p \boxtimes \mathbb{Z}_p$ nonzero such that $(\bar{a},\bar{b})(\bar{c},\bar{d}) = (\bar{0},\bar{0})$. Note then that $(\bar{c},\bar{d})$ is also a zero-divisor and by Affirmation 3, it follows that $\bar{c} \neq \bar{0}$ and $\bar{d} \neq \bar{0}$. From this equality there follows $(\overline{ac - bd}, \overline{ad + bc}) = (\bar{0},\bar{0})$ and thus we obtain

$$\overline{ac - bd} = \bar{0} \qquad (1)$$

$$\overline{ad + bc} = \bar{0} \qquad (2)$$

Multiplying equation (1) by $\bar{a}$, equation (2) by $\bar{b}$ and summing yields $\overline{(a^2 + b^2)c} = \bar{0}$. Since $\bar{c} \neq \bar{0}$, we conclude that $\overline{a^2 + b^2} = \bar{0}$.

Conversely, since $(\bar{0},\bar{0}) \neq (\bar{a},\bar{b}) \in \mathbb{Z}_p \boxtimes \mathbb{Z}_p$ and $\overline{a^2 + b^2} = \bar{0}$, then $(\bar{0},\bar{0}) \neq (\bar{b},\bar{a}) \in \mathbb{Z}_p \boxtimes \mathbb{Z}_p$ and $(\bar{a},\bar{b})(\bar{b},\bar{a}) = (\bar{a}\bar{b} - \bar{b}\bar{a}, \overline{a^2 + b^2}) = (\bar{0},\bar{0})$. That is, $(\bar{a},\bar{b})$ is a zero-divisor. ■

Having characterized the zero-divisors, we also ask how many of them are there, in terms of the prime p. If we denote the cardinality of this set as $|Div(\mathbb{Z}_p[i])|$, then, according to the lists provided by the program (see Table I), it can be seen that for $p = 5$ we have $|Div(\mathbb{Z}_5[i])| = 8 = 2(5) - 2$; for $p = 13$ it follows that $|Div(\mathbb{Z}_{13}[i])| = 24 = 2(13) - 2$; and for $p = 17$, that $|Div(\mathbb{Z}_{17}[i])| = 32 = 2(17) - 2$. This same pattern holds for all odd primes that are a sum of two squares, less than 100. This allows us to conjecture that $|Div(\mathbb{Z}_p[i])| = 2(p-1)$, which is formally proven below.

Affirmation 5. If p is an odd prime which is a sum of two squares, then $|Div(\mathbb{Z}_p[i])| = 2(p-1)$.

Proof. Let $(\bar{a},\bar{b}) \in \mathbb{Z}_p \boxtimes \mathbb{Z}_p$ be a zero-divisor. Then $\bar{a}$ and $\bar{b}$ are nonzero and $\overline{a^2 + b^2} = \bar{0}$. This implies that $\overline{b^2} = \overline{-a^2}$ and by Proposition 1, we have that for all $\bar{a} \in \{\bar{1}, \bar{2}, \ldots, \overline{p-1}\}$:

$$\left(\frac{-a^2}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{a^2}{p}\right)$$

$$= \left(\frac{-1}{p}\right).1$$

$$= 1.$$

Which signifies that $-a^2$ is a quadratic residue modulo p. Equivalently, $\overline{-a^2}$ has two square roots in $\mathbb{Z}_p$ and they are $\bar{b}$ and $\overline{-b}$. Moreover, $\bar{b}$ and $\overline{-b}$ are different because otherwise you would have $\overline{2b} = \bar{0}$, which leads to $\bar{b} = \bar{0}$, which is a contradiction.

In conclusion, if $(\bar{a},\bar{b}) \in \mathbb{Z}_p \boxtimes \mathbb{Z}_p$ is a zero-divisor, the possibilities for $\bar{a}$ are $p - 1$ and for $\bar{b}$ are 2. That is, there exist $2(p-1)$ zero-divisors. ■

Definition 4. An element $(\bar{a},\bar{b}) \in \mathbb{Z}_p \boxtimes \mathbb{Z}_p$ is called nilpotent if there exists $n \in \mathbb{N}$ such that $(\bar{a},\bar{b})^n = (\bar{0},\bar{0})$. We will say that $(\bar{a},\bar{b})$ is 2 −nilpotent if $(\bar{a},\bar{b})^2 = (\bar{0},\bar{0})$.

The Python program in Figure 4 is designed to find the 2-nilpotent elements for each prime p and at the same time indicate the number of 2-nilpotents in each case.

```
def simulacionCZp(p):
    b=[0,0]
    nilp=0
    lista=list(range(0,p))
    for i in range(0,p):
        for j in range(0,p):
            x=[lista[i],lista[j]]
            y=[lista[i],lista[j]]

            z=[(x[0]*y[0]-x[1]*y[1])%p,\
               (x[0]*y[1]+x[1]*y[0])%p]

            if z==b:
                print(x,"*",y,":",z)
                nilp=nilp+1

    print("lista nilpotentes")
    print("numero de elementos 2-nilpotentes:",nilp)
```

Fig. 4. Python program for the 2-nilpotent elements in $\mathbb{Z}_p[i]$.

The values obtained for $p = 5, 13, 17, 29,$ and 37 are given in Table II. Note that in all cases, there is only one 2-nilpotent

element, the trivial $(\bar{0},\bar{0})$. This is also observed for the other odd primes that are sums of two squares and are less than 100. So, we can conjecture that only the null element of $\mathbb{Z}_p[i]$ is $2-$nilpotent, which is proved below.

TABLE II
2-NILPOTENT ELEMENTS IN $\mathbb{Z}_p[i]$, FOR $p = 5, 13, 17, 29,$ AND 37

Primo	Lista Nilpotentes	Total Lista
p = 5	[0, 0] * [0, 0] = [0, 0]	1
p = 13	[0, 0] * [0, 0] = [0, 0]	1
p = 17	[0, 0] * [0, 0] = [0, 0]	1
p = 29	[0, 0] * [0, 0] = [0, 0]	1
p = 37	[0, 0] * [0, 0] = [0, 0]	1

Affirmation 6. Let p be an odd prime which is a sum of two squares. Then, the only $2-$nilpotent element of $\mathbb{Z}_p \boxtimes \mathbb{Z}_p$ is $(\bar{0},\bar{0})$.

Proof. If $(\bar{a},\bar{b})$ is $2-$nilpotent, then $(\bar{a},\bar{b})(\bar{a},\bar{b}) = (\bar{0},\bar{0})$. That is, $(\overline{a^2-b^2}, \overline{ab+ba}) = (\bar{0},\bar{0})$ and we obtain the system

$$\overline{a^2-b^2} = \bar{0} \quad (3)$$

$$\overline{2ab} = \bar{0} \quad (4)$$

Equation (4) implies that $\bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$. Replacing either of the two options in equation (3) yields $\bar{b} = \bar{0}$ or $\bar{a} = \bar{0}$, respectively. In conclusion, $\bar{a} = \bar{0}$ and $\bar{b} = \bar{0}$. ■

Definition 5. An element $(\bar{a},\bar{b}) \in \mathbb{Z}_p \boxtimes \mathbb{Z}_p$ is called idempotent if $(\bar{a},\bar{b})^2 = (\bar{a},\bar{b})$.

It is clear that $(\bar{0},\bar{0})$ and $(\bar{1},\bar{0})$ are idempotents of $\mathbb{Z}_p \boxtimes \mathbb{Z}_p$, which are called trivial idempotents.

The Python program in Figure 5 was designed to show the idempotent elements for each prime p and at the same time indicate the number of these elements.

```
def simulacionCZp(p):
    idemp=0
    lista=list(range(0,p))
    for i in range(0,p):
        for j in range(0,p):
            x=[lista[i],lista[j]]
            y=[lista[i],lista[j]]

            z=[(x[0]*y[0]-x[1]*y[1])%p,\
               (x[0]*y[1]+x[1]*y[0])%p]


            if x==y and x==z:
                idemp=idemp+1
                print(x,"*",y,":",z)

    print("idempotentes:",idemp)
```

Fig. 5. Python program for the idempotent elements in $\mathbb{Z}_p[i]$.

The results given by the above program for $p = 5, 13, 17, 29, 37,$ and 41 are shown in Table III. It can be observed that the nontrivial idempotents for $p = 5$ are $(\bar{3},\bar{1})$ and $(\bar{3},\bar{4})$. In both cases the first component is $\bar{3}$, $\bar{3} = \overline{\left(\frac{5+1}{2}\right)}$ and furthermore $\overline{3^2+1^2} = \bar{0}$ and $\overline{3^2+4^2} = \bar{0}$. For $p = 13$ they are $(\bar{7},\bar{4})$ and $(\bar{7},\bar{9})$. In both cases the first component is $\bar{7}$, $\bar{7} = \overline{\left(\frac{13+1}{2}\right)}$ and moreover $\overline{7^2+4^2} = \bar{0}$ and $\overline{7^2+9^2} = \bar{0}$. This same pattern is observed in the other primes in the table and in all other odd primes less than 100 that are the sum of two squares. Thus, we conjecture that $(\bar{a},\bar{b})$ is a nontrivial idempotent if, and only if, $\bar{a} = \overline{\left(\frac{p+1}{2}\right)}$ and $\overline{a^2+b^2} = \bar{0}$. This is proved below.

TABLE III
IDEMPOTENT ELEMENTS IN $\mathbb{Z}_p[i]$, FOR $p = 5, 13, 17, 29, 37,$ AND 41

Primo	Lista Idempotentes	Total Lista
p = 5	[0, 0] * [0, 0] = [0, 0] [1, 0] * [1, 0] = [1, 0] [3, 1] * [3, 1] = [3, 1] [3, 4] * [3, 4] = [3, 4]	4
p = 13	[0, 0] * [0, 0] = [0, 0] [1, 0] * [1, 0] = [1, 0] [7, 4] * [7, 4] = [7, 4] [7, 9] * [7, 9] = [7, 9]	4
p = 17	[0, 0] * [0, 0] = [0, 0] [1, 0] * [1, 0] = [1, 0] [9, 2] * [9, 2] = [9, 2] [9, 15] * [9, 15] = [9, 15]	4
p = 29	[0, 0] * [0, 0] = [0, 0] [1, 0] * [1, 0] = [1, 0] [15, 6] * [15, 6] = [15, 6] [15, 23] * [15, 23] = [15, 23]	4
p = 37	[0, 0] * [0, 0] = [0, 0] [1, 0] * [1, 0] = [1, 0] [19, 3] * [19, 3] = [19, 3] [19, 34] * [19, 34] = [19, 34]	4
p = 41	[0, 0] * [0, 0] = [0, 0] [1, 0] * [1, 0] = [1, 0] [21, 16] * [21, 16] = [21, 16] [21, 25] * [21, 25] = [21, 25]	4

Recall that if e is a nontrivial idempotent in a ring with unity 1, then e is a zero-divisor since one has $e(e-1) = 0$.

Affirmation 7. Let p be an odd prime which is the sum of two squares and $(\bar{a},\bar{b}) \in \mathbb{Z}_p \boxtimes \mathbb{Z}_p$. Then $(\bar{a},\bar{b})$ is a nontrivial idempotent if, and only if, $\bar{a} = \overline{\left(\frac{p+1}{2}\right)}$ and $\overline{a^2+b^2} = \bar{0}$.

Proof. If $(\bar{a},\bar{b})$ is a nontrivial idempotent, then $(\bar{a},\bar{b})$ is a zero-divisor and by Affirmation 3, $\bar{a} \neq \bar{0}$ and $\bar{b} \neq \bar{0}$. The following system of equations results:

$$\overline{a^2 - b^2} = \bar{a} \quad (5)$$

$$\overline{2ab} = \bar{b} \quad (6)$$

From Equation (6), one has $\bar{2}\bar{a} = \bar{1} = \overline{p+1} \Longrightarrow \bar{a} = \overline{\left(\frac{p+1}{2}\right)}$. Substituting in Equation (5) we obtain

$$\begin{aligned} \overline{a^2 + b^2} &= \overline{2a^2 - a} \\ &= \overline{2\left(\frac{(p+1)^2}{4}\right) - \left(\frac{p+1}{2}\right)} \\ &= \overline{\left(\frac{p+1}{2}\right) - \left(\frac{p+1}{2}\right)} \\ &= \bar{0}. \end{aligned}$$

On the other hand, let $(\bar{a}, \bar{b}) \in \mathbb{Z}_p \boxtimes \mathbb{Z}_p$ with $\bar{a} = \overline{\left(\frac{p+1}{2}\right)}$ and $\overline{a^2 + b^2} = \bar{0}$. Then, $\overline{b^2} = \overline{-a^2}$ and moreover,

$$\begin{aligned} (\bar{a}, \bar{b})^2 &= \left(\overline{a^2 - b^2}, \overline{2ab}\right) \\ &= \left(\overline{2a^2}, \bar{b}\right) \\ &= \left(\overline{2\left(\frac{(p+1)^2}{4}\right)}, \bar{b}\right) \\ &= \left(\overline{\left(\frac{p+1}{2}\right)}, \bar{b}\right) \\ &= (\bar{a}, \bar{b}). \end{aligned}$$

That is, $(\bar{a}, \bar{b})$ is idempotent. ■

For the number of nontrivial idempotent elements, it suffices to look at Table III. The results indicate that independently of the prime p, exactly two nontrivial idempotents are always found. That is, if we denote the cardinality of this set by $\left|Id(\mathbb{Z}_p[i])\right|$, then $\left|Id(\mathbb{Z}_p[i])\right| = 2$. This will be proved below.

Affirmation 8. If p is an odd prime which is a sum of two squares, then $\left|Id(\mathbb{Z}_p[i])\right| = 2$.

Proof. Let $(\bar{a}, \bar{b}) \in \mathbb{Z}_p \boxtimes \mathbb{Z}_p$ be a nontrivial idempotent. Then $\bar{a} = \overline{\left(\frac{p+1}{2}\right)}$ and $\overline{a^2 + b^2} = \bar{0}$. Thus, $\bar{a}$ has a fixed value that depends on p and $\bar{b}^2 = \overline{-a^2}$. By Proposition 1 and the proof of Affirmation 7, $\overline{-a^2}$ has two different square roots in $\mathbb{Z}_p$. That is, $\bar{b}$ takes two different values, which implies that $\left|Id(\mathbb{Z}_p[i])\right| = 2$. ■

To finish this work, it remains to study the invertible elements of $\mathbb{Z}_p[i]$.

Definition 6. An element $\bar{a} + \bar{b}i \in \mathbb{Z}_p[i]$ is invertible if there is a $\bar{c} + \bar{d}i \in \mathbb{Z}_p[i]$ with $(\bar{a} + \bar{b}i)(\bar{c} + \bar{d}i) = 1$.

The Python program in Figure 6 was designed to display the invertible elements for each prime p and at the same time indicate the number of these elements.

```
def simulacionCZp(p):
    b=[1,0]
    unos=0

    lista=list(range(0,p))
    for i in range(0,p):
        for j in range(0,p):
            for k in range(0,p):
                for l in range(0,p):
                    x=[lista[i],lista[j]]
                    y=[lista[k],lista[l]]

                    z=[(x[0]*y[0]-x[1]*y[1])%p,\
                       (x[0]*y[1]+x[1]*y[0])%p]

                    if z==b:
                        unos=unos+1
                        print(x,"*",y,":",z)

    print("Inversos:",unos)
```

Fig. 6. Python program for the invertible elements in $\mathbb{Z}_p[i]$.

The results obtained by the above program for $p = 5$ and $p = 13$ are observed in Tables IV and V. We can observe that the only invertible pairs $(\bar{a}, \bar{b})$ are those with $\overline{a^2 + b^2} \neq \bar{0}$. The same result is observed for all odd primes less than 100 that are the sum of two squares. The proof of this is analogous to that presented in [9, 10], we include it for completeness of this paper and as a benefit to the reader.

TABLE IV
INVERTIBLE ELEMENTS IN $\mathbb{Z}_5[i]$

p = 5	
Total inversos = 16	
[0, 1] * [0, 4] = [1, 0]	[2, 2] * [4, 1] = [1, 0]
[0, 2] * [0, 2] = [1, 0]	[2, 3] * [4, 4] = [1, 0]
[0, 3] * [0, 3] = [1, 0]	[3, 0] * [2, 0] = [1, 0]
[0, 4] * [0, 1] = [1, 0]	[3, 2] * [1, 1] = [1, 0]
[1, 0] * [1, 0] = [1, 0]	[3, 3] * [1, 4] = [1, 0]
[1, 1] * [3, 2] = [1, 0]	[4, 0] * [4, 0] = [1, 0]
[1, 4] * [3, 3] = [1, 0]	[4, 1] * [2, 2] = [1, 0]
[2, 0] * [3, 0] = [1, 0]	[4, 4] * [2, 3] = [1, 0]

Affirmation 9. Let p be an odd prime which is the sum of two squares and $\bar{a} + \bar{b}i \in \mathbb{Z}_p[i]$. Then, $\bar{a} + \bar{b}i$ is invertible if, and only if, $\overline{a^2 + b^2} \neq \bar{0}$.

Proof. If $\overline{a^2 + b^2} = \bar{0}$, by Affirmation 4 we have that $\bar{a} + \bar{b}i$ is a zero-divisor and thus it is not invertible.

On the other hand, if $\overline{a^2+b^2} \neq \bar{0}$, then it is easy to see that the multiplicative inverse of $\bar{a}+\bar{b}i$ is $(\bar{a}+\bar{b}i)^{-1} = (\overline{a^2+b^2})^{-1}(\bar{a}-\bar{b}i) \in \mathbb{Z}_p[i]$. ■

TABLE V
INVERTIBLE ELEMENTS IN $\mathbb{Z}_{13}[i]$

p = 13	
Total inversos = 144	
[0, 1] * [0, 12] = [1, 0]	[1, 7] * [6, 10] = [1, 0]
[0, 2] * [0, 6] = [1, 0]	[1, 9] * [10, 1] = [1, 0]
[0, 3] * [0, 4] = [1, 0]	[1, 10] * [4, 12] = [1, 0]
[0, 4] * [0, 3] = [1, 0]	[1, 11] * [8, 3] = [1, 0]
[0, 5] * [0, 5] = [1, 0]	[1, 12] * [7, 7] = [1, 0]
[0, 6] * [0, 2] = [1, 0]	[2, 0] * [7, 0] = [1, 0]
[0, 7] * [0, 11] = [1, 0]	[2, 1] * [3, 5] = [1, 0]
[0, 8] * [0, 8] = [1, 0]	[2, 2] * [10, 3] = [1, 0]
[0, 9] * [0, 10] = [1, 0]	[2, 4] * [4, 5] = [1, 0]
[0, 10] * [0, 9] = [1, 0]	[2, 5] * [5, 7] = [1, 0]
[0, 11] * [0, 7] = [1, 0]	….....
[0, 12] * [0, 1] = [1, 0]	[12, 6] * [7, 3] = [1, 0]
[1, 0] * [1, 0] = [1, 0]	[12, 7] * [7, 10] = [1, 0]
[1, 1] * [7, 6] = [1, 0]	[12, 9] * [3, 1] = [1, 0]
[1, 2] * [8, 10] = [1, 0]	[12, 10] * [9, 12] = [1, 0]
[1, 3] * [4, 1] = [1, 0]	[12, 11] * [5, 3] = [1, 0]
[1, 4] * [10, 12] = [1, 0]	[12, 12] * [6, 7] = [1, 0]

As for the number of invertible elements, the program shows in Tables IV and V, that for $p = 5$ there are 16 invertibles and for $p = 13$ there are 144 invertibles. We note then that $16 = (5-1)^2$ and $144 = (13-1)^2$. Moreover, the same can be observed for the other odd primes less than 100 that are the sum of two squares. Thus, if we denote the cardinality of this set by $|Inv(\mathbb{Z}_p[i])|$, then $|Inv(\mathbb{Z}_p[i])| = (p-1)^2$. This is proved below.

Affirmation 10. If p is an odd prime which is the sum of two squares, then $|Inv(\mathbb{Z}_p[i])| = (p-1)^2$.

Proof. According to Affirmations 4 and 9, it can be concluded that every nonzero element $\bar{a}+\bar{b}i \in \mathbb{Z}_p[i]$, is a zero-divisor or invertible: this depends on whether $\overline{a^2+b^2} = \bar{0}$ or $\overline{a^2+b^2} \neq \bar{0}$, respectively. That is, the set $\mathbb{Z}_p[i]$ is partitioned into three classes, the zero-divisors, the invertibles, and the zero element. By Affirmation 5, $p^2 = |Inv(\mathbb{Z}_p[i])| + 2(p-1) + 1$, which implies that $|Inv(\mathbb{Z}_p[i])| = p^2 - 2(p-1) - 1 = (p-1)^2$, which is what we wanted to prove. ■

IV. CONCLUSIONS

We present below the most relevant aspects that emerged during the development of this work and at the same time we would like to make some recommendations to continue with this study, providing new elements for discussion and research.

- The process employed in this work allows using computational programming to conjecture results in algebra. This shows that programming is not only useful in engineering or applied sciences but also allows interesting computational studies in algebra, an abstract area. Consequently, many of the known processes in mathematical problem-solving using programming were evidenced, such as: problem understanding, exploration, case study, program design and implementation, desktop testing, and evaluation.

- We emphasize the pedagogical importance of the process followed in this work to obtain the results. This allows, through computer programming, to obtain results, which in turn lead to a differentiated mental development in terms of the observation of patterns, formulation of hypotheses, and finally the formal proofs of the assertions. In the same way, other mental processes are developed in the student as a consequence of the deep understanding of the set being studied together with its structure, the programming of the different algebraic concepts, analysis of the results, observation of patterns, formulation of hypotheses, and their theoretical proof.

- The computer programs developed in this work can be modified to study other important elements in a ring such as nilpotent in general, regular, associated, and irreducible, among many others [5, 8, 19]. In this case, one could also consider various rings of integers modulo n or some subclasses as $\mathbb{Z}_p, \mathbb{Z}_{p^\alpha}, \mathbb{Z}_{pq}$ with p, q primes and the corresponding complexification of each of them. Even other sets of integers modulo n such as Eisenstein, Hurwitz and Lipschitz integers could be considered [23, 24].

- As a continuation of this work and also relying on computational programming, additional studies on the group of invertible elements, its generators, and the cardinality of this set can be considered. As for the zero-divisors we can observe that the simulations found can be used to determine which and how many are the pairs $(\bar{c}, \bar{d})$ such that $(\bar{a}, \bar{b})(\bar{c}, \bar{d}) = (\bar{0}, \bar{0})$, where $(\bar{a}, \bar{b})$ is a fixed zero-divisor. That is, in the language of graphs we would be thinking about determining which and how many vertices are connected to the given vertex $(\bar{a}, \bar{b})$. This would lead to the study of the zero-divisor graph of the ring $\mathbb{Z}_p[i]$ with p an odd prime of the form $4k+1$ or more generally of the rings $\mathbb{Z}_n[i]$ and $\mathbb{Z}_n$ [8]. Finally, as for idempotents, this paper shows that in the case of the rings $\mathbb{Z}_p[i]$ with p an odd prime of the form $4k+1$, only two nontrivial idempotents result, which does not allow us to go deeper into this ring. However, when considering more general rings such as $\mathbb{Z}_n$, $\mathbb{Z}_n[i]$ and even quaternions modulo n, a nontrivial number of idempotents arise [20]. This makes viable a deeper study of them in terms of characterization, cardinality, classes of these, associated ordered set, and all notions arising from this order [19].

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