Directional Dependence via Copulas: Examples and Applications in Engineering

Dependencia direccional mediante cópulas: ejemplos y aplicaciones en ingeniería

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Abstract- In engineering practice, risk rarely arises from isolated variables acting independently. More often, critical failures emerge when extreme conditions occur simultaneously, revealing dependence patterns that cannot be captured by traditional correlation measures. In such contexts, the relationship between variables may intensify or weaken in specific regions of their distributions, particularly in the upper or lower tails. This behavior, commonly referred to as directional dependence, has direct implications for reliability assessment and design decisions. Copula functions offer a flexible framework to model these asymmetric dependence structures while allowing marginal behaviors to be treated separately. This study presents a structured overview of directional dependence via copulas, describing its theoretical basis, estimation procedures, and practical interpretation. Two applied examples are developed to illustrate the approach: joint extreme precipitation in hydraulic engineering and concurrent wind- seismic loading in structural engineering. The results highlight how ignoring tail dependence can lead to systematic underestimation of joint risk. To support transparency and reproducibility, the study includes implementation guidelines in R and graphical representations that facilitate practical adoption in engineering analysis.

Index Terms— **Copula models; Directional dependence; Engineering applications; Risk analysis; Tail dependence.**

***Resumen*- En la práctica ingenieril, el riesgo rara vez surge de variables aisladas que actúan de manera independiente. Con mayor frecuencia, las fallas críticas se producen cuando condiciones extremas coinciden, revelando patrones de dependencia que no pueden describirse adecuadamente mediante medidas tradicionales de correlación. En estos escenarios, la relación entre variables puede intensificarse o modificarse en regiones específicas de sus distribuciones, particularmente en las colas superiores o inferiores. Este comportamiento, conocido como dependencia direccional, tiene implicaciones directas en la evaluación de la confiabilidad y en las decisiones de diseño.**

Las cópulas ofrecen un marco flexible para modelar estas estructuras de dependencia asimétrica, permitiendo analizar de forma separada el comportamiento marginal y la estructura conjunta. En este trabajo se presenta una explicación estructurada de la dependencia direccional vía cópulas, abordando sus fundamentos teóricos, procedimientos de estimación e interpretación práctica. Se desarrollan dos aplicaciones: precipitaciones extremas en ingeniería hidráulica y cargas combinadas viento–sismo en ingeniería estructural. Los resultados muestran que ignorar la dependencia en colas puede conducir a una subestimación sistemática del riesgo conjunto. Además, se incluyen lineamientos de implementación en R y representaciones gráficas que facilitan la reproducibilidad y su aplicación en contextos ingenieriles.

Palabras claves— **Modelos de cópula; Dependencia direccional; Aplicaciones en ingeniería; Análisis de riesgo; Dependencia en colas.**

I. INTRODUCTION

Engineering systems exposed to natural or operational hazards rarely fail due to isolated variables acting independently. Instead, critical conditions often arise from the concurrent behavior of multiple extreme variables. Classical correlation measures, although useful for average association, are insufficient to describe how variables interact under extreme scenarios. In particular, extreme co-occurrence patterns may differ substantially from central dependence structures, requiring more flexible modeling approaches. In traditional engineering analyses, the relationship between variables is commonly modeled under assumptions of independence, normality, or symmetric correlation. However, multiple studies have shown that in real-world scenarios, especially in the presence of extreme events, variables exhibit asymmetric relationships that cannot be adequately captured by classical measures such as Pearson or Spearman correlation [1 -5].

It is important to emphasize that Pearson and Spearman correlation coefficients are not always suitable for

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characterizing all forms of dependence between variables. The Pearson coefficient measures the strength of the linear relationship between continuous variables; its interpretation is straightforward, but it relies on the assumption that the data follow an approximately normal joint distribution and a strictly linear relationship. Moreover, it is highly sensitive to the presence of outliers, which can significantly distort the measure of association. Therefore, its use is more appropriate when the variables are continuous, the relationship is clearly linear, and there are no major deviations from normality [6].

By contrast, the Spearman coefficient is calculated from data ranks, making it less sensitive to extreme values and independent of the normality assumption. This coefficient can detect monotonic associations—whether increasing or decreasing—even when they are not strictly linear, making it a more flexible alternative in scenarios where the relationship between variables cannot be assumed to be linear [6].

This asymmetry is known as directional dependence, a phenomenon that becomes particularly relevant in disciplines such as hydraulic, structural, and energy engineering, where systems are exposed to extreme conditions and combined risks **[7-10].** For example, in hydraulic engineering, the simultaneous occurrence of intense rainfall in neighboring basins can lead to critical flooding. Recent studies apply copulas to estimate this extreme dependence, better capturing the joint behavior of high precipitation events [11]. In structural engineering, it has been shown that wind and seismic loads may coincide, affecting the safety of sensitive infrastructures [12]. Additionally, applications have been reported in finance, hydrology, structural reliability, energy systems, and environmental sciences. These fields benefit from copula models to characterize joint risk, extreme co-occurrence, and complex multivariate dependence structures [13 -18].

The development of copula theory has enabled the construction of joint models capable of representing nonlinear and asymmetric dependence patterns. By separating marginal behavior from the dependence mechanism, copulas allow engineers to capture extreme co-movement structures that traditional multivariate models often overlook [19], [20].

Despite theoretical advances, the adoption of copulas with a directional focus in applied engineering remains limited. This article seeks to reduce that gap by providing an accessible explanation of the fundamentals of directional dependence via copulas [20], presenting two practical examples (in hydraulic and structural engineering), and including R code [21] and high-resolution graphics to ensure replicability.

II. THEORETICAL FOUNDATIONS

The analysis of dependencies between variables is essential in engineering, particularly in contexts where phenomena exhibit extreme or nonlinear behaviors. Traditionally, measures such as Pearson correlation have been employed to evaluate the relationship between variables, but these tools present limitations as they fail to adequately capture asymmetric relationships or tail dependencies. Copulas emerge as a powerful alternative, allowing the separate modeling of marginal distributions and the joint dependence structure between random variables [20], [22]. This section introduces the theoretical framework required to understand copulas, Sklar's theorem as their foundation, tail dependence coefficients, and the main copula families applied in engineering.

A. Sklar's Theorem

Sklar's theorem provides the theoretical foundation for copula modeling by establishing that any multivariate distribution can be decomposed into its marginal distributions and a function that encodes their dependence structure. This linking function, known as a copula, connects uniformly transformed marginals. When the marginal distributions are continuous, the associated copula is uniquely determined. This result enables engineers to model marginal behavior and dependence structure separately, offering significant flexibility in practical applications.

$$H(x,y) = C\big(F_X(x), F_Y(y)\big) \tag{1}$$

with marginal distributions F_X and F_Y (see [20], [23]). Equation (1) shows the formal definition of copulas.

B. Definition of Copulas

A copula $C: [0,1]^2 \to [0,1]$ (bivariate case) is a joint distribution function of two random variables uniformly distributed on [0,1]. Formally, it must satisfy:

i) $C(u,0) = C(0,u) = 0$ for all $u \in [0,1]$.

ii) $C(u,1) = C(1,u) = u$ for all $u \in [0,1]$.

iii) C is 2-increasing: for any rectangle $[u_1,u_2] \times [v_1,v_2] \subset [0,1]^2$,

$$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0.$$

These properties guarantee that C is a valid distribution with uniform marginals. Moreover, any continuous joint distribution can be reconstructed from its marginals and a copula via Sklar's theorem [20], [22].

C. Upper and Lower Tails and Directional Dependence Coefficient

Tail dependence describes the persistence of association under extreme realizations. Rather than measuring average correlation, it evaluates whether one variable remains extreme given that the other variable has already exceeded a high or low threshold. This feature becomes particularly relevant in risk-oriented engineering analyses. Upper tail: both variables take high values simultaneously. Lower tail: both variables take low values simultaneously.

The tail dependence coefficients are defined as:

$$\lambda_L = \lim_{u \to 0^+} P(Y \leq F_Y^{-1}(u) \mid X \leq F_X^{-1}(u)), \qquad (2)$$
$$\lambda_U = \lim_{u \to 1^-} P(Y > F_Y^{-1}(u) \mid X > F_X^{-1}(u)). \qquad (3)$$

These coefficients capture behaviors that linear correlation does not reflect, and their theoretical estimation depends on the chosen copula. For instance, certain copulas exhibit upper-tail dependence, others lower-tail dependence, and some both or none. **Equations (2) and (3) show the formal definition of each of the directional dependencies** [22], [23].

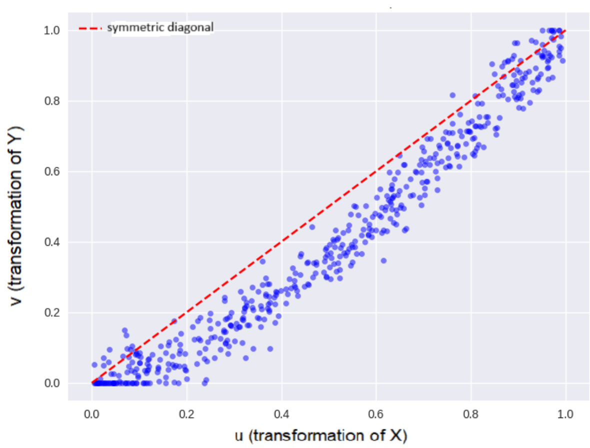

Fig 1: Symmetric dependence modeled by the Student-t copula.

In Figure 1, a scenario of directional dependence is illustrated, where variable Y strongly depends on variable X after transformation to the uniform scale $[0,1]$. As X increases, values of Y tend to concentrate around the diagonal, though with greater dispersion in the lower-left region. The dashed red line represents perfect symmetry ($Y = X$), highlighting the asymmetry in the scatter: variability in Y is greater when X takes small values, while for large values of X both variables align more closely. This pattern reflects how the underlying copula captures stronger upper-tail dependence than lower-tail dependence, a central aspect in directional dependence analysis for engineering and risk applications.

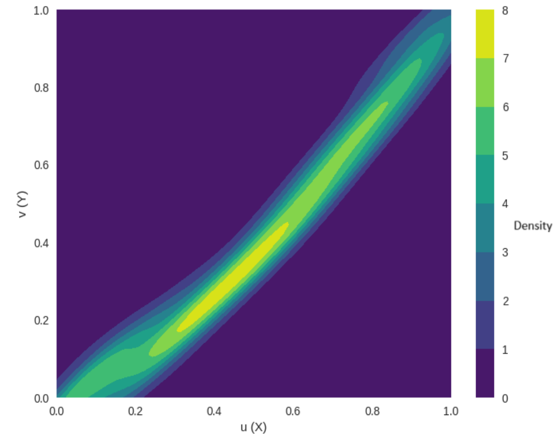

Figure 2: Density contours under directional dependence.

In Figure 2, the joint density contours of the transformed variables $u(X)$ and $v(Y)$ under a copula model with directional dependence are shown. A clear concentration of density along the diagonal reflects strong positive association between the variables. However, the elongated and asymmetric contour shape reveals that dependence is not perfectly symmetric: association intensifies as the variables take high values, i.e., in the upper tail of the distribution. This pattern is characteristic of scenarios where extreme events tend to occur simultaneously, a key aspect in the assessment of joint risks in hydraulic, structural, or environmental engineering.

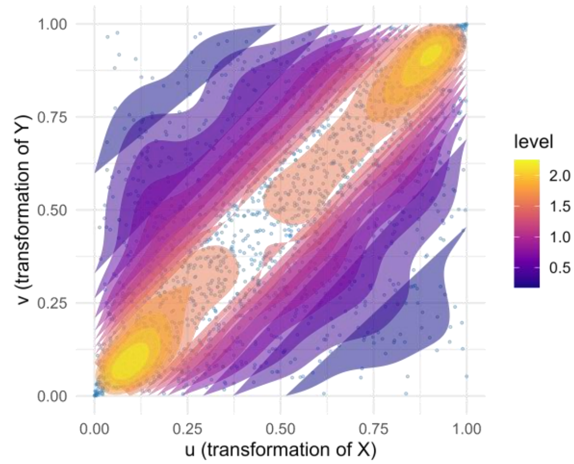

Fig 3: Symmetric dependence captured by a Student-t copula.

Figure 3 represents a case of symmetric dependence, characteristic of the Student's t copula, where association between variables is balanced across both tails of the distribution. This implies that extremely low and extremely high values tend to occur jointly with similar probabilities. Unlike asymmetric copulas such as Gumbel or Clayton, which emphasize a single tail, the symmetry of the t copula ensures that dependence intensity is evenly distributed across the data range. The plot shows densities concentrated along both upper and lower diagonals, reflecting this symmetric correspondence. This property is particularly useful in engineering and finance, where extreme risks may manifest as severe losses or exceptional gains, requiring models that fairly capture both scenarios [20], [24].

C. Types of Copulas

Several copula families are commonly employed depending on the type of dependence structure observed in the data. The Gaussian copula, derived from the multivariate normal distribution, captures symmetric dependence but does not generate tail dependence unless correlation approaches unity. The Student's t copula extends this framework by incorporating heavy tails controlled by its degrees of freedom parameter, allowing symmetric upper and lower tail dependence.

Archimedean copulas provide asymmetric alternatives. The Gumbel copula emphasizes upper-tail dependence, making it suitable for modeling joint high extremes such as extreme rainfall or maximum structural loads. Conversely, the Clayton copula captures lower-tail dependence, which is useful when simultaneous low-level failures or minima are of primary concern. Selecting the appropriate family therefore depends on whether extreme behavior is concentrated in one tail, both, or neither [11], [12], [22] and [23]. In summary, the choice of copula must align with the type of directional dependence relevant to the engineering system under study whether emphasizing high extremes, low extremes, both, or none.

III. MATERIALS AND METHODS

To estimate copula-based models and capture directional dependence between variables, the methodology commonly involves the following steps:

A. Parametric Estimation via Maximum Likelihood

The maximum likelihood estimation (MLE) method assumes a parametric family of copulas, together with marginal distributions for each variable, and estimates the parameters that maximize the joint likelihood of the transformed data [25], [26]. Specifically, the marginal data are first transformed to uniform scales using the empirical (or estimated) cumulative distribution function. The copula density function is then applied to construct the likelihood, which is maximized with respect to the dependence parameters.

B. Semiparametric and Nonparametric Methods

When one does not wish to fully assume the form of the copula distribution, or when extreme data are scarce, semiparametric or nonparametric estimators can be employed to relax strong assumptions. For example, methods based on inference functions for margins (IFM), minimum divergence estimation (Alpha-Divergence), or estimation using local probit transformations have shown good performance under weak dependence and moderate sample sizes [25], [27].

C. Model Selection: AIC, BIC, and Goodness-of-Fit

Once parameters have been estimated for several candidate copula families, models are compared using information criteria such as AIC or BIC. Goodness-of-fit (GOF) tests and graphical evaluations are also applied, including density comparisons, QQ-plots of transformations, and contour plots of the estimated versus empirical copula density [26], [28].

D. Tools and Software in R

For practical implementation, the use of the most up-to-date R packages is recommended, such as:

Copula: for standard fitting of Gaussian, Student's t, and Archimedean copulas.

VineCopula: useful for pair-copula constructions, modeling complex dependencies, and computing upper/lower tail dependence.

CopBasic: for empirical estimation, tail dependence functions, and diagnostic plots.

These packages allow parameter estimation via MLE, model comparison, and computation of theoretical and empirical tail dependence coefficients. They also facilitate the generation of high-resolution plots for contour visualization, density estimation, and diagnostic analysis.

IV. RESULTS

This section summarizes the main findings through two applied examples and their broader implications. Example 1 examines hydraulic engineering with joint extreme precipitation, while Example 2 focuses on structural engineering with seismic–wind load dependence. Building on these cases, we discuss the implications for design and reliability and outline key limitations and challenges of copula-based modeling, highlighting both its potential and constraints in engineering practice.

The procedures adopted in the hydraulic and structural examples are summarized in Figures 4 and 5. The diagrams trace the progression from raw data treatment to dependence modeling and final engineering interpretation, making explicit the sequence of analytical decisions involved in each case.

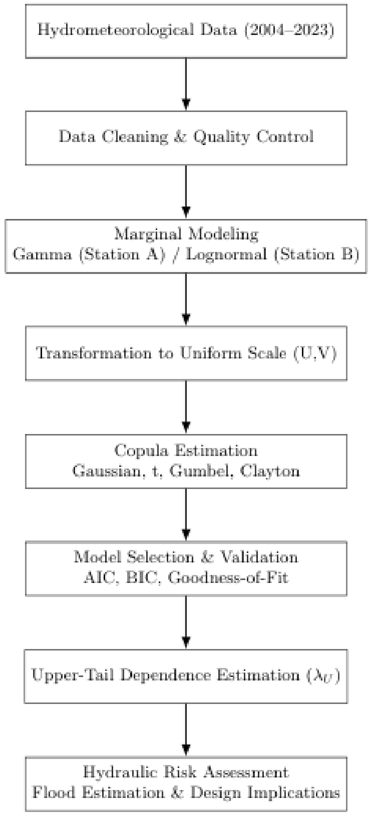

Figure 4: Methodological workflow for directional dependence modeling in hydraulic engineering applications.

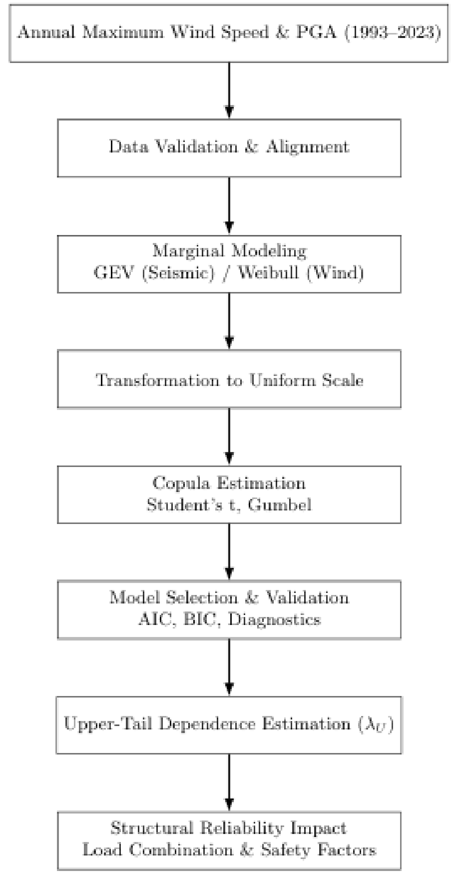

Figure 5: Methodological workflow for directional dependence modeling in structural engineering applications.

A. Example 1: Hydraulic Engineering – Joint Extreme Precipitations

1. Data and Hypotheses

For this example, we consider daily series of precipitation intensity (rainfall per unit of time) collected at two nearby rain gauge stations in a mountain basin over a 20-year period (assuming clean data with minimal missing values). The variables analyzed are:

X: Daily precipitation intensity at Station A (mm/day)
Y: Daily precipitation intensity at Station B (mm/day)

Our specific interest is to estimate upper-tail dependence (λ_U), that is, the probability that both stations record very high precipitation simultaneously, given that one is already in the upper percentiles.

2. Procedure

Data cleaning: remove days with missing values and verify consistency and homogeneity.

Marginal transformation: fit marginal distributions to each series (e.g., Gamma, Log-normal, or Generalized Pareto for extreme values), then transform the marginal data using the empirical distribution functions $F_X(x)$ and $F_Y(y)$ to obtain uniform variables on [0,1]: $U = F_X(x)$, $V = F_Y(y)$.

Fitting candidate copulas: test at least Gaussian, Student's t, Gumbel, and Clayton copulas using maximum likelihood estimation.

Calculation of λ_U (and optionally λ_L): compute for each fitted copula using theoretical formulas when available, and empirical estimation via upper quantiles (e.g., 95th or 99th percentile) of the transformed data.

Model comparison: select the copula with the best fit according to AIC/BIC and graphical evaluation (joint density contours, QQ-plots, etc.).

Visualization: scatterplots of U, V, joint density plots, upper-tail contours, etc.

3. Estimated Results and Model Comparison

Based on recent studies, such as "Copula-based dependency modelling of hydraulic" [29], the Gumbel copula often captures upper-tail dependence more effectively in regions with extreme rainfall, while the Student's t copula provides flexibility when both tails exhibit mild dependence [29].

Suppose the fitted results are:
Gaussian copula: $\lambda_U \approx 0$.
Student's t copula (df = 4): $\lambda_U = 0.25$.
Gumbel copula (parameter = 2): $\lambda_U = 0.3$.
Clayton copula: $\lambda_U = 0$ (no upper-tail dependence)

The best model according to AIC/BIC was the Gumbel copula, followed by the Student's t, indicating that upper-tail dependence is significant for this basin.

4. Implications

These results imply that hydraulic designs such as drainage systems, sewer networks, and flood-control structures must account for the probability of simultaneous high precipitation at multiple stations, since the actual risk may be much higher than estimated under weak dependence or symmetric models.

Moreover, as a cautionary note: if only a Gaussian copula is considered, runoff volumes or system capacity requirements could be underestimated during critical events.

B. Example 2: Structural Engineering – Dependence Between Seismic and Wind Loads

1. Data and Hypotheses

In this example, we analyze the dependence between two types of extreme loads affecting structures: seismic load (e.g., maximum response to ground accelerations) and extreme wind load (high wind speeds). Suppose we have historical series of seismic records (ground intensity, PGA, or equivalent measures) and extreme wind records (annual maximum wind speed) for the same locality or region over at least 30 years.

The objective is to estimate how likely both loads are to be simultaneously high, and how this dependence affects safety factors and structural designs such as walls, frames, and load-bearing systems.

2. Procedure

Selection of historical data: validate quality, clean missing values, and ensure seismic and wind records coincide in time and geography.

Marginal transformation: fit appropriate marginal distributions (e.g., Generalized Extreme Value (GEV) for seismic maxima, Log-normal or Weibull for wind, or empirical distributions if required). Then transform each variable to the uniform scale using the marginal cumulative distribution: $U = F_{\text{seismic}}(x), V = F_{\text{wind}}(y)$.

Fitting candidate copulas: at least a Student's t copula (allowing dependence in both tails) and a Gumbel copula (capturing upper-tail dependence). Parameters are estimated via maximum likelihood.

Calculation of directional dependence coefficients: λ_U (upper tail) as the main measure, and optionally λ_L.

Model selection: compare candidates using AIC, BIC, and graphical/numerical goodness-of-fit (GOF) tests, QQ-plots, joint density contours, and assessment of extreme values.

Evaluation of structural impact: given a structural design model, compute design loads considering the estimated dependence between wind and seismic forces; for example, assess whether current codes underestimate combined loads when assuming independence or weak dependence.

4. Estimated Results and Model Comparison

A recent study, "Hybrid Bayesian-Copula-based damage probability of tall buildings under concurrent seismic and strong wind" [30], found that tall buildings may face significantly increased damage risk when upper-tail dependence between strong wind and seismic load is considered, compared to models treating them as independent [30].

Suppose the fitted results are:

Gaussian copula: $\lambda_U \approx 0.05$.
Student's t copula (df = 5): $\lambda_U = 0.20$.
Gumbel copula (parameter = 2.5): $\lambda_U = 0.28$.
In this scenario, with the Gumbel copula preferred by

AIC/BIC, design load factors incorporating simultaneous wind and seismic effects suggest that structural safety margins should be increased—for example, by enhancing seismic capacity design or revising local codes—to withstand these combined events.

5. Implications for Structural Design

Considering directional dependence between extreme loads has important implications:

- Structural design codes may underestimate risk when assuming independent extreme loads.
- Higher safety factors or more conservative designs may be required for structural elements.
- In seismic zones with strong winds (storms, hurricanes, cold fronts), combined load specifications should be reviewed.
- It may affect the sizing of critical elements, component connections, and performance under fatigue or damage during multiple extreme events.

D. Implications for Design and Reliability

The inclusion of directional dependence in probabilistic analysis is not merely a theoretical refinement but a practical necessity. Ignoring this dependence, particularly in extreme tails, can lead to a significant underestimation of actual risk, severely affecting the design, safety, and expected lifespan of critical infrastructures and systems.

For example, in geotechnical reliability, [31] demonstrated that the choice of different copulas can yield markedly different failure probability estimates; using a Gaussian copula when tail dependence is strong may underestimate system failure probability by several orders of magnitude. In another study on structural reliability, it was found that assuming independence or using an inadequate copula can result in errors ranging from 4 to 10 times in failure probability estimates compared to models that correctly capture extreme event dependence [32].

The practical effects of such underestimation include:
- Structural designs with insufficient load factors, compromising safety against concurrent extreme events (wind, seismic, combined loads).
- Hydraulic infrastructures unable to tolerate simultaneous high events from multiple sources, increasing the risk of critical failures such as floods or overflows.
- Unexpected maintenance or rehabilitation costs due to damage greater than anticipated under models that ignore directional dependence.
- Revision of design codes: local regulations may require explicit incorporation of tail dependence to ensure adequate reliability levels.
- Improved estimates of risk, expected loss, service life, and performance under extreme events, enabling more informed mitigation strategies.

In conclusion, adopting models that capture directional dependence allows:
- More accurate estimation of joint probabilities of extreme events.
- Avoidance of structural or hydraulic surprises when simultaneous extreme conditions occur.
- Design, financing, and risk mitigation decisions based on a more robust quantitative foundation.

D. Limitations and Challenges

While modeling directional dependence through copulas offers multiple advantages, it also faces important practical challenges that must be acknowledged when applied in engineering. The most relevant are described below.

1. Scarcity of Extreme Data

In many cases, analyses depend on the availability of sufficient data in the distribution tails—extremely high or low values. When extreme data are scarce, tail dependence estimates (λ_U, λ_L) exhibit high variance and substantial bias. Missing data, imprecise measurements, or unreliable records during extreme events exacerbate the problem. Bayesian models or inference methods with resampling (bootstrap) can partially alleviate scarcity but require additional assumptions [33].

2. Selection of Appropriate Copulas

Choosing a suitable copula family is essential: using a symmetric copula when the phenomenon exhibits tail asymmetry may lead to risk underestimation, while employing a heavy-tailed copula without supporting data may cause overfitting. Furthermore, some common copulas do not allow dependence in both tails, limiting their applicability to certain engineering systems. Selection criteria (such as AIC, BIC, GOF), together with graphical evaluation and cross-validation, become critical [33], [34].

3. Scalability to Higher Dimensions

When the number of variables increases (e.g., multiple stations, multiple load types, numerous climatic variables), copula modeling faces the curse of dimensionality. Classical multivariate copulas (or even vine copulas) involve a large number of parameters, complicating estimation, requiring larger samples, and increasing computational time. Sparse, truncated, or composite likelihood-based models are among current strategies to address this limitation [34], [35].

4. Other Practical Challenges

- Verification of continuity or accuracy assumptions for marginals, since errors in marginal estimation propagate to the joint model.
- Treatment of heteroscedasticity or structural changes in historical records, which may affect model consistency.

- Interpretation and integration into engineering codes and local regulations, which often do not explicitly account for dependence in extreme events.
- Computational costs, particularly for Monte Carlo simulations, bootstrap resampling, complex vine models, or other nonparametric methods in high dimensions.

In summary, although directional dependence via copulas is a powerful tool, its application must be undertaken with awareness of these limitations. Engineers should verify the robustness of estimates, perform validations, and, when necessary, adopt adaptive strategies to ensure reliable results.

V. DISCUSSION

The characterization of directional dependence through copulas, as presented in this study, reveals that many engineering systems exhibit dependence structures more complex than those captured by traditional measures such as Pearson or Spearman correlation. The applications developed for extreme precipitation and seismic–wind loads consistently show that tail dependence, particularly in the upper tail, is far more relevant for risk assessment than average dependence. This finding aligns with recent studies warning that linear or symmetric models tend to smooth the joint probability of extreme events, leading to insufficient estimates of actual risk in physical and environmental systems.

In both examples analyzed, models based on asymmetric copulas particularly the Gumbel copula provided a better fit to empirical patterns of extreme dependence. This result reinforces the idea that when phenomena are governed by physical processes favoring the simultaneity of high values (regional storms, interaction between load mechanisms, compound effects), dependence symmetry is not a realistic property. In contrast, the Gaussian copula proved incapable of reproducing extreme behaviors, confirming observations in the literature that identify it as unsuitable for scenarios where tails dominate risk.

An important observation is that directional dependence has not only statistical but also engineering implications. In the hydraulic case, the magnitude of upper-tail dependence directly affects the design of drainage infrastructure, reservoir capacity, and flood management plans. When dependence is strong, the probability of simultaneous extreme precipitation increases considerably, which may lead to critical underestimations of runoff volume if models assuming independence are used. Similarly, in structural engineering, the presence of dependence between wind and seismic loads contradicts traditional independence assumptions employed in many design codes, implying potential deficiencies in safety factors and performance evaluation under compound events.

The discussion also highlights that the added value of copulas is not limited to statistical modeling but extends to physical interpretations of dependence. In many contexts, tail directionality can be linked to underlying geophysical or mechanical mechanisms: large-scale storms affecting neighboring basins simultaneously, structural interaction under concurrent excitations, or correlations induced by regional atmospheric or seismic processes. In this sense, the copula framework enables integration of statistical theory with expert knowledge of the system.

Nevertheless, the examples presented also reveal methodological challenges. In particular, tail dependence estimation is sensitive to the availability of extreme data, which is often limited in climatic and engineering records. Moreover, although this study demonstrates the effective use of bivariate copulas, extension to multivariate scenarios requires more complex models, such as vine copulas or hierarchical structures, whose computational cost and risk of overfitting can be significant. These aspects suggest that adopting more robust model selection methods (GOF, alpha-divergence, cross-validation, bootstrap) is indispensable for real applications.

Finally, this study emphasizes that directional dependence should become a more prominent tool in modern risk analysis and engineering. Explicitly incorporating dependence structures rather than assuming them allows for more realistic estimates, improved infrastructure design, and reduced probability of failure under extreme conditions. This opens an opportunity for technical standards to evolve, including more flexible models grounded in statistical evidence, particularly in countries vulnerable to multi-hazard phenomena.

VI. CONCLUSIONS

The analysis of directional dependence through copulas represents a significant advancement in modeling extreme phenomena in engineering. Throughout this work, it has been shown that copulas allow precise and flexible representation of dependence structures between variables exhibiting nonlinear or asymmetric behaviors, particularly in the tails of their distributions.

The examples presented in hydraulic and structural engineering demonstrate that assuming independence or symmetric dependence between variables can lead to considerable underestimation of the risk of concurrent extreme events. Appropriate copula selection, especially those capturing upper-tail dependence such as the Gumbel copula, or symmetric dependence such as the Student’s t copula substantially improves risk estimates and design decisions.

Furthermore, the relevance of computational tools such as the **copula** and **VineCopula** packages in R has been highlighted, as they facilitate the practical implementation of the approach. These resources make it possible to apply sophisticated models without the need to develop algorithms from scratch, democratizing access to these methods for engineering professionals.

Although limitations exist such as scarcity of extreme data and challenges in high-dimensional multivariate modeling the benefits of adopting copula-based models far outweigh the difficulties when protecting critical infrastructures against extreme events.

Consequently, it is recommended that engineers and risk modelers:

- Incorporate copula analysis into reliability and design studies.
- Train technical teams in these tools, leveraging support from open-source software.
- Promote the inclusion of explicit tail dependence in design codes and technical regulations in multivariable contexts.

The systematic use of copulas with a directional focus has the potential to significantly raise standards of safety, efficiency, and resilience across multiple engineering disciplines, aligning modern design with practices based on robust statistical evidence [36], [37].

Future work may focus on extending directional copula models to higher-dimensional settings, particularly in systems where multiple interacting variables influence performance simultaneously. It would also be valuable to embed these dependence structures within reliability-based design approaches, allowing for a more consistent integration between probabilistic modeling and engineering decision-making. From a hydraulic perspective, additional research could examine compound extreme events under non-stationary conditions, especially in contexts shaped by climate variability and long-term environmental change.

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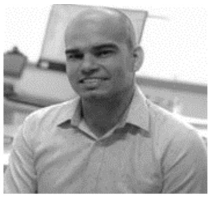

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