Conduction of fluids through porous parallel walls

Conducción de fluidos a través de paredes paralelas porosas

Diana Marcela Devia Narvaez, Elisabeth Restrepo Parra[2], Fernando Mesa
[1] *Ph.D. Docente, Grupo de Investigación: Laboratorio de física del plasma (LAFIP), Facultad de Ciencias Básicas, Universidad Tecnológica de Pereira, Risaralda, Colombia.*
dmdevian@utp.edu.co
[2] *Ph.D. Ingeniera Electricista, Grupo de Investigación: Laboratorio de física del plasma (LAFIP), Facultad de Ingenierías, Universidad Nacional de Colombia-sede Manizales, Caldas, Colombia.*
erestrepopa@unal.edu.co
[3] *M.Sc. Docente, Grupo de Investigación: Laboratorio de física del plasma (LAFIP), Facultad de Ciencias Básicas, Universidad Tecnológica de Pereira, Risaralda, Colombia.*
femesa@utp.edu.co

***Resumen*— Una conducción de fluido se inyecta continuamente o expulsada a través de un par de paredes porosas paralelas y se escapa en ambas direcciones a lo largo del canal. El flujo forma un punto de estancamiento en el centro y la emanación está restringido por un campo magnético. Un análisis teórico de las soluciones de estado estacionario de las ecuaciones MHD en el caso incompresible se da como una función de tres parámetros: el número de Reynolds Re, el número de Reynolds magnético Rm y Alfvenic número de Mach MA para algunos de los límites asintóticos significativas.**

Palabras clave— Conducción de fluidos, expulsión, inyección, paredes porosas paralelas

***Abstract*—A conducting fluid is continuously injected or ejected through a pair of parallel porous walls and it escapes in both directions along the channel. The flow forms a stagnation point at the center and the effluence is restricted by a magnetic field. A theoretical analysis of steady state solutions of the MHD equations in the incompressible case is given as a function of three parameters: the Reynolds number Re, the magnetic Reynolds number Rm and Alfvenic Mach number MA for some of significant asymptotic limits.**

***Key Word* —conducting fluids, Ejection, Injection, Parallel porous walls.**

I. INTRODUCTION

The bidimensional problem of a viscous and incompressible fluid in a porous channel with a stagnation point in the center was initially studied by Berman [1] whose work was motivated to give a model that explained the separation of uranium from U_{238} to U_{235} by gaseous diffusion. The uranium is previously turned to the gas UF_6, which has appropriate characteristics for its manipulation. In this pioneering work the problem of the stationary case was solved, using similar solutions to reduce from the Navier-Stokes equation to a differential equation of fourth degree, with a pair of border conditions in each wall. Berman found analytical solutions for the asymptotic situation of low Reynolds numbers in the case of suction in the walls. Later authors have studied different physical situations from this problem, Sellars [2], Yuang [3], Proudman [4], Shrestha [5], Terril [6], Brady and Acrivos [7], Brady [8], Robinson [9], Zaturska *et. al*. [10], Watson et al [11], Cox [12], Banks [13, 14] that in general has treated, for example the cases of symmetrical, asymmetric flows, walls with acceleration, different speeds from suction or injection in the walls superior and inferior.

In this work a conducting fluid which is continuously injected or ejected through a pair of parallel porous walls and which escapes in both directions along the channel is study.

II. BASIC EQUATIONS OF THE MAGNETOHYDRODYNAMICS PROBLEM

The Navier-Stokes and the Ohm law, equations could be written in a reduced form as:

$$(\partial_t - \nu\nabla^2)\nabla^2\xi - \left[\xi,\nabla^2\xi\right] - \frac{1}{4\pi\rho}\left[\nabla^2\psi,\psi\right] = 0 \qquad (1)$$

$$(\partial_t - \nu_m\nabla^2)\psi - \left[\xi,\psi\right] = cE_z \qquad (2)$$

In the last equations, the velocity and the magnetic field are given by the following equations:

Fecha de Recepción: 08 de Septiembre de 2013
Fecha de Aceptación: 25 de Junio de 2014

$$V_x = \partial_y \xi \ , \ V_y = -\partial_x \xi \ , \ B_x = \partial_y \psi \ , \ B_y = -\partial_x \psi \quad (3)$$

The bracket $[f, g] = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$ defines the Jacobian of f and g functions, additionally $\nu_m = \frac{c}{4\pi\sigma}$ is the magnetic diffusion, $-\nabla^2\xi$ is the z component of the vorticity $\boldsymbol{W} = \nabla \times \boldsymbol{V}$, ψ is the z component of the potential vector $\mathbf{A}$ ($\nabla \times \mathbf{A} = \boldsymbol{B}$), which means, the J_z component can be written in terms of the ψ function like $-\nabla^2\psi = \frac{1}{4\pi} J_z$. The brackets $\left[\xi, \nabla^2\xi\right]$ and $[\xi, \psi]$ represent the convection transport terms of $-\omega_z$ and ψ_z, respectively. Additionally, the bracket $\left[\nabla^2\psi, \psi\right]$ represents the curl z component of the Lorentz force. We Suppose that there is an invariance of the translation in z, this is, for example, $\frac{\partial E_x}{\partial x} = 0$ and $\frac{\partial E_y}{\partial y} = 0$. Besides $E_z = E_z(t)$, just depends on the time. In general the electrical field can be described by $\boldsymbol{E} = -\nabla\varphi + \boldsymbol{E}_z(t) - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$, therefore it could be found a potential $\hat{\varphi}$, such that $\hat{\varphi} = \varphi(x, y, t) + E(t)z$. We will study a model where the lateral walls, which are supposed to be distant in the z direction, cannot be charged electrically, then the z - component of the electrical field E_z is zero. This fact will permit us to see ahead, the use of similar solutions for uncoupling the equations (1) and (2). It can be supposed as well that those walls are conductive but they are in short-circuiting for a lab model (see fig. 1). Finally, these walls could be in the infinite, this last case is presented for instance in an astrophysic model. From above it can be deduced that in a tree-dimensional model the boundary conditions in z are related to the E_z electrical field.

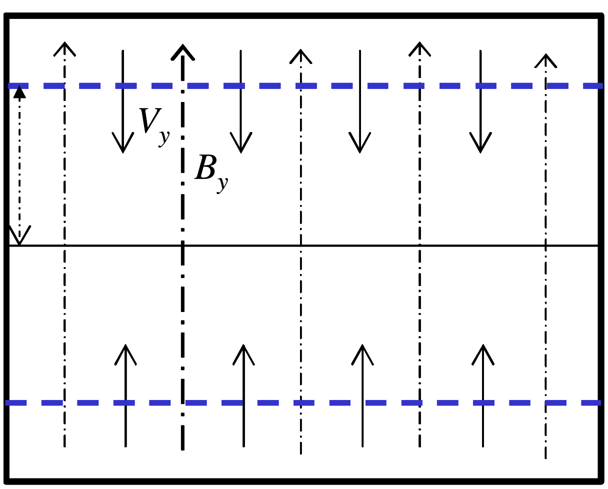

Figure 1. A conductor fluid injected through the walls. The magnetic field is represented without interaction between the field and the fluid.

The equations system (1) and (2) expounded previously, admits in general similar solutions of the form:

$$\xi = xf(y, t, \eta) + g(y, t, \eta), \quad (4)$$

$$\psi = xp(y, t, \eta) + q(y, t, \eta), \quad (5)$$

Where η represents all parameters involved, in this case, the viscosity, the magnetic diffusivity and external magnetic field. Replacing the equations (4) and (5) in the equations (1) and (2),the following dimensionless equations system is obtained in the suction and injection cases of a fluid between two parallel and porous plates with an external magnetic field:

$$f_{tyy} - \frac{1}{R_e} f_{yyyy} = \left(ff_{yy} - (f_y)^2\right)_y - \frac{1}{M_A^2}\left(pp_{yy} - (p_y)^2\right)_y \quad (6)$$

$$p_t - \frac{1}{R_m} p_{yy} = fp_y - f_y p \quad (7)$$

R_e is the Reynolds number, R_m is the magnetic Reynolds number, H_a is the Hartmann number and M_A is the alfvenic Mach number. Notice that now $f = f(y, t, R_e, R_m, M_A)$ and $p = p(y, t, R_e, R_m, M_A)$. Given R_e, R_m and M_A, the system given by the equations (6) and (7) could be solved numeric or analytically in the asymptotic situation that we will study later. Sometimes, when the shooting technique is used in order to solve cases for the equations system given previously, the problem is invested and the parameters employed to make the calculations are R_e, R_m y M_A. Notice that if $p = 0$ the problem decreases to the pure fluidness case. The general problem so expounded is quiet complex from a mathematical point of view. We will study the case in which a conductor, incompressible and viscous flow, goes in or out through a pair of parallel infinite perforated walls with the same suction rate or with injection in both walls (separated by a distance of $2h_0$). The flow that interacts with a magnetic field is basically perpendicular to the walls in the case of being conductors. If the walls are dielectric the magnetic field can have x and y components in the boundary. This magnetic field is modified by the conductor flow movement, as it is shown in figure 2.

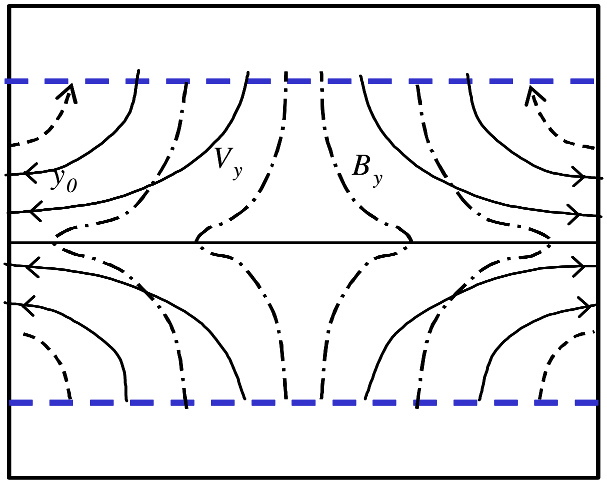

Figure 2. Lines speed and the field in the case where it is considered that interaction with the magnetic field exists. This case comes when injection of flow through the walls exists.

The magnetic field $\mathbf{B}=(B_x,B_y,0)$ and the speed $\mathbf{V}=(V_x,V_y,0)$ could be obtained then of the following form:

$$V_x=\partial_y\xi=xf_x \qquad V_y=-\partial_x\xi=-f \qquad (8)$$

$$B_x=\partial_y\psi=xp_y \qquad B_y=-\partial_x\psi=-p \qquad (9)$$

III. Boundary Conditions

For the case of a fluid that enters or leaves for a pair of perforate and parallels walls, in presence of a magnetic field the initial conditions, are: $f(y,0,\eta)=f(y,\eta)=f_0(y,\eta)+f_1(t)$, here the temporary part $f_1(t)$ and $p(y,0,\eta)=p(y,\eta)=p_0(y,\eta)+p(t)$, here $p_1(t)$ are small interferences of f_0 y p_0, that they in turn are the solutions of the stationary case obtained from the equations (1) and (2). Additionally η is determined by a fixed parameters set of the system v_0, v_{0m} and B_0 which correspond to the values in the cinematic and magnetic viscosities and in the magnetic fields respectively. Here it is convenient to define the following operators:

$$Hf=\left[f(-1),f_y(-1),f(1),f_y(1)\right] \qquad (10)$$

$$H_Sf=\left[f(1),f_y(1),f(0),f_{yy}(0)\right] \qquad (11)$$

$$H_Af=\left[f(1),f_y(1),f_y(0),f_{yyy}(0)\right] \qquad (12)$$

$$Kp=\left[p(-1),p(1)\right] \qquad (13)$$

$$K_\beta p=\left[p(1),p_y(1)\right] \qquad (14)$$

The boundary condition for the velocity, the condition over f, in the case of suction in the walls is $Hf=[1,0,-1,0]$ and for the injection case $Hf=[-1,0,1,0]$.

Since that in our numerical calculus we have integrated the equations (6) and (7) between the half of the channel width ($y=0$) and in the wall ($y=1$), we define the operators H_S and H_A which correspond to the cases of symmetric solutions, $H_S=[\pm1,0,0,0]$, for injection (+) and suction (-)) and antisymmetric ($H_A=[\pm1,0,0,0]$ for injection (+) and suction (-), respectively. Note that if f represents a symmetric flow then this should be an odd function so that $f(0)=0$, and therefore the origin is always a stagnation point.

The boundary conditions for the magnetic field, the conditions over p, depend on the walls and flow conductor character. Nevertheless the following condition for both suction and injection cases, should be generally satisfied:

$$K_Ap=[-1,\beta] \qquad (15)$$

If the walls are conductors the constant β is adjusted depending on the characteristic parameters of the problem. For example, in the low viscosity and high conductivity regimes $\beta=0$ is taken. Since in this case the flow drags the magnetic field lines so that they become parallel and therefore the magnetic field and the velocity satisfy the same boundary condition over the wall. This last condition $\beta=0$, is valid for all the conductor walls. If the walls are dielectric then the magnetic field between them, is supposed to be originated by a pair of external coils that generate an external magnetic field in the form:

$$B_x^e=cx \qquad B_y^e=-cy+b \qquad (16)$$

It means that the function ψ that represents the magnetic field flow (equation (2)) is now given by the expression:

$$\psi^{ext}=-bx+cxy \qquad (17)$$

Providing that the magnetic field in the walls can take any value, the boundary conditions for the magnetic field inside the walls that permit the couple with the external magnetic field, can be briefly written in the following way:

$$K_\alpha p=\left[p_y(1),p_y(0)\right] \qquad (18)$$

If $K_\alpha = [c,0]$, the symmetric case of dielectric walls is obtained, but if K_α=[0,0], then the walls will be metallic. Since in this work we just present the symmetric flow case, the condition $f(\pm 1) = \mp 1$ is fixed for the flow and for the magnetic field $p(\pm 1) = -1$ and for the numerical case, what means to take $R_e > 0$ and $R_m > 0$ for the suction case while for the injection case $R_e < 0$ y $R_m < 0$, without varying the boundary conditions. Additionally when using the above convention, the time changes of sign in the injection case. It is clear that the negative time and negative Reynolds number definitions do not have any physical interpretation, it is just a mathematic artifice used in this kind of problems in order to facilitate the numerical calculus. In some cases it is convenient to use an integration of the equation (6). For the stationary case the equations system given in the equations (6) and (7) is described by the following equations system:

$$-\frac{1}{R_e} f_{yyy} = C + \left(f f_{yy} - (f_y)^2 \right) - \frac{1}{M_A^2}\left(p p_{yy} - (p_y)^2 \right) \quad (19)$$

$$-\frac{1}{R_m} p_{yy} = f p_y - f_y p \quad (20)$$

The constant C of integration is determined starting from the values in the boundary, that in stationary case of conductor walls and of injection of flowing in the walls, C is given by the equation:

$$C = \left\{ -\frac{1}{R_e} f_{yyy} - \left(f_{yy} \right) + \frac{1}{M_A^2}\left(p_{yy} - \beta^2 \right) \right\}_{y=1} \quad (21)$$

This integration constant C, on the other hand is directly related with the pressures gradient according to the x axis through the expression:

$$(\partial p / \partial x = -ax - xP'^2 / 4\pi) \quad (22)$$

What is to say the pressures gradient depends not only on the x magnetic field component but on the position according to the x axes.

IV. Asymptotic Approximations R_E <<1 and R_M <<1.

In the injection case with low Reynolds numbers, the magnetic field lines are now rigid just by a small perturbation which is caused by the flow movement, this one at the same time is very viscous for this limit (R_e<<1). Such magnetic field can be written in the following way:

$$p = p_0 + p_1, \quad (23)$$

where p_0 is the field value that we assumed as constant and by simplification reasons can be taken the same as the unit. On the other hand, p_1 is a small perturbation which as it was previously said, it is caused by the fluid movement. Thus the equations (19) and (20) previously linearized, can be written in the following way: (see fig. 2),

$$-\frac{1}{R_e} f''' = C - \frac{1}{M_A{}^2} p_0 p_1'' \quad (24)$$

$$\frac{1}{R_m} p_1'' = f' p_0 \quad (25)$$

In the expressions above the second order terms have been suppressed, that means, we have taken the first two terms of the expansion $p = 1 + R_m p_1 + \ldots$.

On the other side, the term $1/R_m$ is very big, but p_1'' is very small, so the equation (25) is valid. When replacing the equation (24) the following differential equation is obtained:

$$f''' = -CR_e + H_a{}^2 p_0{}^2 f' \quad (26)$$

This equation at the same time has as solution (with p_0=1):

$$f = \left(y - \frac{Senh(H_a y)}{H_a Cosh(H_a)} \right) \Big/ \left(1 - \frac{Tanh(H_a)}{H_a} \right) \quad (27)$$

and consequently replacing the equation (25) the following expression for p_1 is obtained:

$$p_1 = -\frac{R_m((H_a y^2/2) - Cosh(H_a y)/H_a Cosh(H_a))}{H_a - Tanh(H_a)} + D \quad (28)$$

Here, H_a is the Hartmann number defined previously. Also the integration constant D is calculated keeping in mind that the wall interference should be null, and then it remains defined like:

$$D = \frac{R_m (H_a/2 - 1/H_a)}{H_a - Tanh(H_a)} \quad (29)$$

Figure 3 shows the velocity component behavior according to the x axis direction for different values of the Hartmann number. Note that when the Hartmann number grows , that means, the magnetic field becomes stronger (M_A<<1), the fluid behavior is similar to the Hartmann flow where the velocity is constant at the center of the channel and it strongly varies when is near the walls until diminishing to zero exactly over the wall.

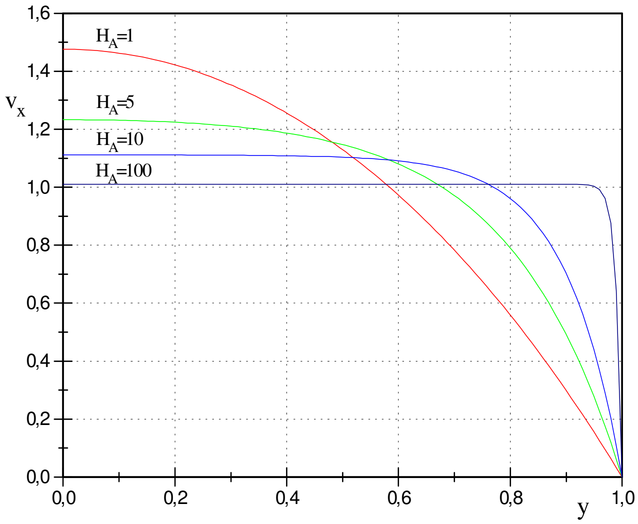

Figure 3. Speed profile and their behavior for several values of the Hartmann number. It is observed that when Ha>>1 appears a limit layer in the wall.

On the other hand, taking into account the boundary condition in the wall $f(y=1)=1$, it is found that the constant C is related to the other constants through the following formula:

$$C = \frac{H_a^{\ 3}}{R_e}\left(\frac{1}{H_a - \tanh(H_a)}\right) \qquad (30)$$

Figure 4 shows the relation between the constants C, H_a and R_e given in the equation (30). Additionally if $H_a >> 1$ (for example, $M_A << (R_e R_m)^{(1/2)} << 1$), it implies that $C \approx H_a^{\ 2} / R_e$, so that it can be deduced that it should exist a strong gradient that moves the fluid outside. The magnetic field roughness controls then the fluid movement, avoiding it to leave. On the other hand if $H_a << 1$, the magnetic field lines are "less rigid " and in this case the condition $CR_e \approx 3$ is satisfied, thus the viscous effects are now the ones that control the fluid movement.

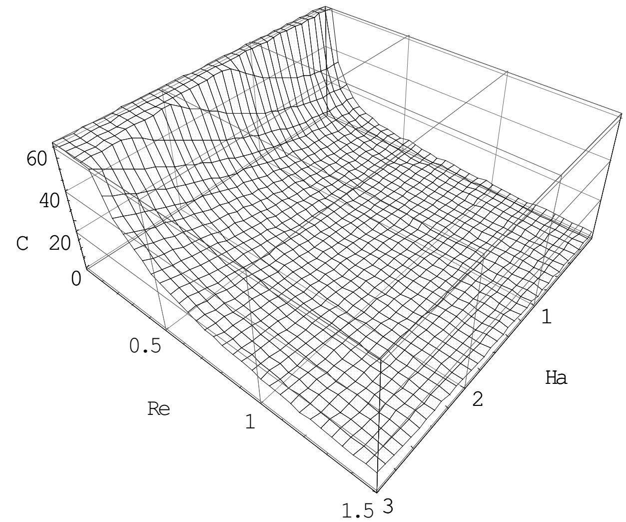

Figure 4. Relation between C, Ha and Re in the asymptotic case of Rm<<1 and Re<<1.

On the other hand, figure 5 shows the current lines and the magnetic field obtained by the basic equations numerical integration. Note that the rigidity of the magnetic field lines, as well as the component B_x in the wall are not annulled. In this graphic it is difficult to see, due to the scale that it was built with.

Figure 6 illustrates the speed and field profiles, where the appearance of the limit layer before mentioned is shown. Similarly, how it was made in the previous asymptotic case, the solutions obtained upon being integrated numerically the equations (19) and (20) for the Runge-Kutta method, for the values $R_e = 0.1$, $R_m = 0.1$, $M_A = 0.3$ and $C = 31.3254$, they coincide with the obtained through the equation (30), where the value that is obtained is $C = 31.3326$. So it is shown again a good agreement between the asymptotic results and found numerals upon integrating the complete equations system. On the other hand, in the numeric integration that was made for several Reynolds number values, they do not show appreciable variations, for both profiles of the speed and the magnetic field, in the range $0.1 \le R_e \le 30$.

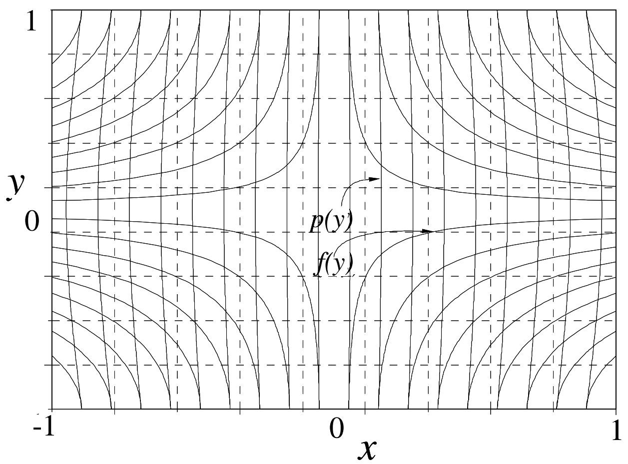

Figure 5. For the asymptotic case, Rm<<1 y Re<<1, the field lines for the speed and the magnetic field is shown. In this case Rm=Re=0.1, MA=0.3 and C=31.3254.

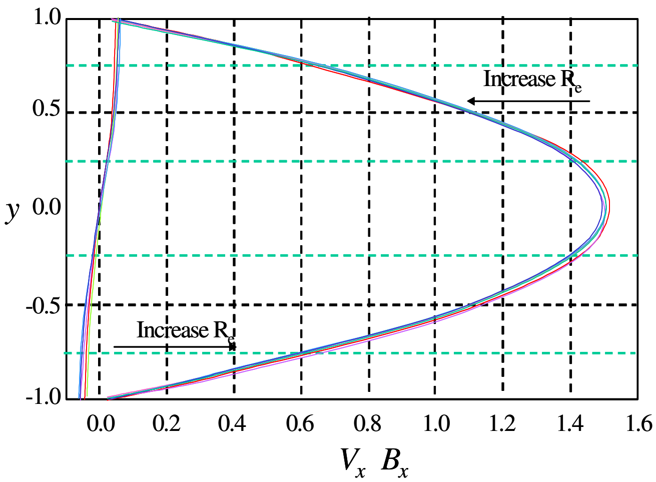

Figure 6. For the case Rm<<1 and Re<<1, the profile of the speed and the magnetic field is shown. It is observed it that the field does is not null in the wall.

REFERENCES

[1] A. S. Berman, "Laminar flow in channels with porous walls," Journal of Applied Physics, vol. 24 pp1232-1235, March 1953.

[2] J. R. Sellars, "Laminar flow in channel with porous walls at high suction Reynolds number," Journal of Applied Physics, vol 26 pp 489-490, Apr 1955.

[3] S. W. Yuang, "Further investigation of laminar flow in channel with porous walls," Journal of Applied Physics vol 27 pp. 267-269,March 1956.

[4] I. Proudman, "An example of steady laminar flow at large Reynolds number," Journal of Fluid Mechanics, vol. 9 pp. 593-602 ,December 1960.

[5] G. M. Shrestha, "Singular perturbation problems of laminar flow in a uniformly porous channel in the presence of a transverse magnetic field," Quarterly Journal of Mechanics and Applied Mathematics, 2nd ed., vol. 20, pp. 233-246 , May 1967.

[6] R. M. Terril," Laminar flow in a uniformly porous channel,", Aeronaut. Quart, vol. 15, pp. 299-310, 1964.

[7] J.F.Brady, A. Acrivos." Steady flow in a channel or tube with an accelerating surface velocity.An exact solution to the Navier Stokes with reverse flow," journal of Fluid Mechanics, vol. 112, pp. 127-150, November 1981.

[8] J. F. Brady, "Flow development in a porous channel and tube," Physics of Fluids, vol. 27, pp. 1061-1067, March 1984.

[9] W. A. Robinson,"The existence of multiple solutions for the laminar flow in a uniformly porous channel with suction at the both walls," Journal of Engineering Mathematics, vol 10, pp 23-40, Apr 1976.

[10] M. B. Zaturska, P. G. Drazin, W. H. H. Banks, "On the flow of a viscous fluid driven along a channel by suction at porous walls," Fluid Dynamics Research, vol. 4, pp.151-178, 1988.

[11] E. B. B. Watson, W. H. H. Banks, M. B. Zaturska, P. G. Drazin,"On transition to chaos in two-dimensional channel flow symmetrically driven by accelerating walls," Journal of Fluid Mechanics, vol. 212, pp. 451-485, March 1990.

[12] S. M. Cox," Two dimensional low of a viscous fluid in a channel with porous walls," Journal of Fluid Mechanics, vol. 227, pp. 1-33,January 1991.

[13] W. H. H. Banks, P. G. Drazin, M. B. Zaturska,"On perturbation of Jeffrey-Hammel flow,"Journal of Fluid Mechanics, vol 186 ,559-581, January1988.

[14] W. H. H. Banks, M. B. Zaturska,"On flow through a porous annular pipe," Physics of Fluids ,vol. 4 ,pp. 1131-1141, Apr 1992.